

# On integer radii coin representations of the wheel graph

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## Abstract

A *flower* is a coin graph representation of the wheel graph. A *petal* of the wheel graph is an edge to the center vertex. In this paper we investigate flowers whose coins have integer radii. For an  $n$ -petaled flower we show there is a unique irreducible polynomial  $P_n$  in  $n$  variables over the integers  $\mathbb{Z}$ , the affine variety of which contains the cosines of the internal angles formed by the petals of the flower. We also establish a recursion that these irreducible polynomials satisfy. Using the polynomials  $P_n$ , we develop a parameterization for all the integer radii of the coins of the 3-petal flower.

**2000 MSC:** 05C10, 05C25, 05C31, 05C35.

**Keywords:** planar graph, coin graph, flower, polynomial ring, Galois theory

## 1 Introduction

By a *coin graph* we mean a graph whose vertices can be represented as closed, non-overlapping disks in the Euclidean plane such that two vertices are adjacent if and only if their corresponding disks intersect at their boundaries, i.e. they touch. For  $n \in \mathbb{N}$  the *wheel graph*  $W_n$  on  $n + 1$  vertices is the simple graph obtained by connecting an additional center vertex to all the vertices of the cycle  $C_n$  on  $n$  vertices. These additional edges are called *petals*. A coin graph representation of a wheel graph is called a *flower*. In Figure 1 we see an example of a flower on the left, and a configuration of coins that does not form a flower on the right.

The study of flowers is central in many discrete geometrical settings, in particular in circle packings [11] and also in the study of planar graphs in general, since every planar graph has a coin graph representation. That a coin graph is planar is clear, but that the converse is true is a nontrivial topological result, usually credited to Thurston [12], but is also due to both Koebe [7] and Andreev [1]. For a brief history of this result we refer to [13, p. 118]. Numerous simply stated, but extremely hard problems involving coin graphs can be found in a recent and excellent collection of research problems in discrete geometry [3]. Also, Brightwell and Scheinerman [4] explored integral representations of coin graphs, where the radii of the coins can take arbitrary positive integer values.

In this paper we study algebraic relations the radii of flowers must satisfy. We first show that for every  $n \geq 3$  the cosines of the central angles of an  $n$ -petal flower are contained in the affine variety of an irreducible polynomial  $P_n$  in  $n$  variables over the integers. We note that the cosines are more

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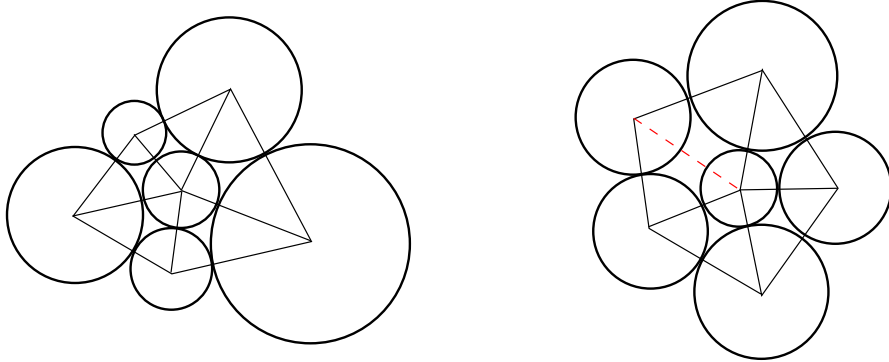


Figure 1: Examples of a flower and a non-flower.

interesting than the sines in this case, for the mere reason that cosines of the angles of an integer-sided triangle are all rational. In particular, for the case  $n = 3$  we then find a parametrization of all integer  $n$ -tuples in this variety of  $P_n$ . Also for the case of  $n = 3$ , we obtain all rational, and hence integer, radii of four mutually tangent circles, sometimes called *Soddy circles* as Frederick Soddy rediscovered Descartes' Circle Theorem in 1936 [2]. Our parametrization differs from the one obtained by Graham et al. in [5] as it is free of any equations relating the parameters.

The rest of the paper is organized as follows: in Section 2 we state our main terminology and definitions. We also present and discuss some basic observations and consequences from the definitions. In Section 3 we use Galois theory to formally define the polynomials  $P_n(x_1, \dots, x_n)$  whose affine variety contains  $(\cos \theta_1, \dots, \cos \theta_n)$  where  $\theta_1, \dots, \theta_n$  are the internal angles of an  $n$ -petaled flower. We then prove our main result of this paper, that each  $P_n$  is an irreducible polynomial over  $\mathbb{Q}$ . In Section 4 we consider the special case of a 3-petal flower. In this case we have four mutually tangent Soddy circles, and we derive a free parametrization of all rational radii of the outer circles when the inner circle has radius one. This will then yield an equation free parametrization of all integer radii of four mutually tangent Soddy circles.

## 2 Definitions, setup and basic informal observations

In what follows  $\mathbb{N} = \{1, 2, 3, \dots\}$  is the set of natural numbers. For  $n \in \mathbb{N}$  we let  $[n] = \{1, \dots, n\}$ . For each  $n \in \mathbb{N}$ , an  $n$ -petal flower imposes a relation on the radii of its coins. For such a flower, assume the radius of the center coin is  $r$  and the radii of the  $n$  outer coins are  $r_1, \dots, r_n$  in clockwise order. We first note that there is an obvious equation relating the  $r_i$ : for each pair of radii  $r_i$  and  $r_{i+1}$  of consecutive petals around a center coin of radius  $r$  we obtain a triangle with sides of length  $r + r_i$ ,  $r + r_{i+1}$ , and  $r_i + r_{i+1}$  and the angle  $\theta_i$  at the center vertex is given by

$$\theta_i = \arccos \left( \frac{(r + r_i)^2 + (r + r_{i+1})^2 - (r_i + r_{i+1})^2}{2(r + r_i)(r + r_{i+1})} \right). \quad (1)$$

The equation that determines a flower with petals of radii  $r_1, \dots, r_n$  is

$$\sum_{i=1}^n \theta_i = 2\pi. \quad (2)$$

For  $G \subseteq S_n$ , a polynomial  $f$  is  $G$ -symmetric if  $f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$  for all  $\sigma \in G$ . We see that (2) is a  $D_n$ -symmetric function in terms of  $r_1/r, \dots, r_n/r$ , where  $D_n$  is the dihedral group of symmetries on the regular polygon with  $n$  sides. In [10] it is shown that for *reflection groups* like the dihedral group  $D_n$  there is a basis of polynomials just like the elementary symmetric functions for the symmetric group  $S_n$ . As we will discuss, if  $x_i = \cos \theta_i$  for each  $i \in [n]$ , then (2) will corresponds to a symmetric polynomial  $f \in \mathbb{Q}[x_1, \dots, x_n]$ . Also, if the center coin has radius  $r = 1$ , and so  $\theta_i = \theta_i(1, r_i, r_{i+1})$  is a function of only the two consecutive radii  $r_i$  and  $r_{i+1}$ , then (2) will corresponds to a  $D_n$ -symmetric polynomial  $g \in \mathbb{Q}[r_1, \dots, r_k]$ . In particular, for general radius  $r$  of the center vertex (replacing  $r_i$  with  $r_i/r$ ), if  $d = \deg(g)$ , which we define as the sum degree, then  $r^d g\left(\frac{r_1}{r}, \dots, \frac{r_n}{r}\right) \in \mathbb{Q}[r, r_1, \dots, r_n]$  is a homogeneous element and

$$r^d g\left(\frac{r_1}{r}, \dots, \frac{r_k}{r}\right) = \sum_{i=0}^d g_i r^i \in \mathbb{Q}[r_1, \dots, r_k][r], \quad (3)$$

where each  $g_i \in \mathbb{Q}[r_1, \dots, r_n]$  is a  $D_n$ -symmetric polynomial. Although intuitively clear, we will in what follows demonstrate this claim informally in an explicit example. To obtain a symmetric function  $f = f(x_1, \dots, x_n)$  we will take the cosine of both sides of (2). Using the relation  $e^{i\theta} = \cos \theta + i \sin \theta$  and then taking the real and imaginary parts of  $e^{i(\theta_1 + \dots + \theta_n)} = e^{i\theta_1} \dots e^{i\theta_n}$ , we obtain the following technical lemma.

**Lemma 2.1** *For  $n \geq 1$  we have the following generalized addition formulae for cos and sin:*

$$\cos\left(\sum_{i=1}^n \theta_i\right) = \sum_{2^{n-1} \text{ terms}} \pm \text{cs}(\theta_1) \text{cs}(\theta_2) \dots \text{cs}(\theta_n),$$

where the sum on the right is taken over the  $2^{n-1}$  possible terms where (i) each cs-function represents either sin or cos and (ii) each term has an even number  $2e$  of sin-functions and the sign of the term is given by  $(-1)^e$ .

Similarly for sin we have

$$\sin\left(\sum_{i=1}^n \theta_i\right) = \sum_{2^{n-1} \text{ terms}} \pm \text{cs}(\theta_1) \text{cs}(\theta_2) \dots \text{cs}(\theta_n),$$

where the sum on the right is taken over the  $2^{n-1}$  possible terms where (i) each cs-function represents either sin or cos and (ii) each term has an odd number  $2e + 1$  of sin-functions and the sign of the term is given by  $(-1)^e$ .

If  $x_i = \cos \theta_i$  for each  $i \in [n]$  then  $y_i = \sin \theta_i$  satisfies the equation  $x_i^2 + y_i^2 = 1$  and hence  $y_i = \pm \sqrt{1 - x_i^2}$ . The geometric properties of the coin graph determine that for the interior angles  $\theta_i$  we have  $0 \leq \theta_i < \pi$  and so  $\sin \theta_i \geq 0$ . Hence we have  $y_i = \sqrt{1 - x_i^2}$  and so both  $\cos \theta_i$  and  $\sin \theta_i$  are in terms of  $x_i$ .

**Definition 2.2** *We define the algebraic expressions  $\text{EC}_n$  and  $\text{ES}_n$  by taking the sine or cosine of (2) and expanding using Lemma 2.1.*

$$\text{EC}_n(x_1, \dots, x_n) = \cos\left(\sum_{i=1}^n \theta_i\right), \quad \text{ES}_n(x_1, \dots, x_n) = \sin\left(\sum_{i=1}^n \theta_i\right).$$

EXAMPLE: For  $n = 1$  we have

$$\text{EC}_1(x_1) = x_1, \quad \text{ES}_1(x_1) = y_1 = \sqrt{1 - x_1^2},$$

and for  $n = 2$  we have

$$\text{EC}_2(x_1, x_2) = x_1x_2 - \sqrt{1 - x_1^2}\sqrt{1 - x_2^2}, \quad \text{ES}_2(x_1, x_2) = x_2\sqrt{1 - x_1^2} + x_1\sqrt{1 - x_2^2}.$$

Directly by the addition formulae for cosine and sine we have the following recursive property of these expressions.

**Lemma 2.3** *For each  $i \in \{1, \dots, n\}$  we have*

$$\begin{aligned} \text{EC}_n(x_1, \dots, x_n) &= x_i \text{EC}_{n-1}(\widehat{x}_i) - y_i \text{ES}_{n-1}(\widehat{x}_i), \\ \text{ES}_n(x_1, \dots, x_n) &= y_i \text{EC}_{n-1}(\widehat{x}_i) + x_i \text{ES}_{n-1}(\widehat{x}_i). \end{aligned}$$

where  $y_i = \sqrt{1 - x_i^2}$  and  $(\widehat{x}_i) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ . In particular for  $i = 1$  we have

$$\begin{aligned} \text{EC}_n(x_1, \dots, x_n) &= x_1 \text{EC}_{n-1}(\widehat{x}_1) - y_1 \text{ES}_{n-1}(\widehat{x}_1), \\ \text{ES}_n(x_1, \dots, x_n) &= y_1 \text{EC}_{n-1}(\widehat{x}_1) + x_1 \text{ES}_{n-1}(\widehat{x}_1). \end{aligned}$$

Note that the expressions  $\text{EC}_n$  and  $\text{ES}_n$  are symmetric in  $x_1, \dots, x_n$ . As informally demonstrated here below, these will yield symmetric polynomials (see [6, p. 252] for more information and general algebraic properties of symmetric polynomials.)

For a fixed  $n \in \mathbb{N}$  (2) yields the algebraic equation  $\text{EC}_n = 1$ . By repeatedly isolating one term that contains a  $y_i$  and squaring, then rearranging the terms, we obtain a polynomial equation  $C_n = 0$ . For example for  $n = 1, 2, 3, 4$  we obtain

$$\begin{aligned} C_1(x_1) &= x_1 - 1 \\ C_2(x_1, x_2) &= (x_1 - x_2)^2 \\ C_3(x_1, x_2, x_3) &= (x_1^2 + x_2^2 + x_3^2 - 2x_1x_2x_3 - 1)^2 \\ C_4(x_1, x_2, x_3, x_4) &= (x_1^4 + x_2^4 + x_3^4 + x_4^4 \\ &\quad - 2(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_1^2x_4^2 + x_1^2x_3^2 + x_2^2x_4^2) \\ &\quad + 4(x_1^2x_2^2x_3^2 + x_2^2x_3^2x_4^2 + x_1^2x_3^2x_4^2 + x_1^2x_2^2x_4^2) \\ &\quad + 4x_1x_2x_3x_4(2 - x_1^2 - x_2^2 - x_3^2 - x_4^2))^2. \end{aligned}$$

Note that for  $n \geq 2$  it appears that  $C_n$  is always a square polynomial, something we will prove in Section 3.

By (1) we have for each  $i \in [n]$

$$x_i = \frac{(r + r_i)^2 + (r + r_{i+1})^2 - (r_i + r_{i+1})^2}{2(r + r_i)(r + r_{i+1})}.$$

Substituting these  $x_i$  into the polynomial equation  $C_n = 0$  yields a rational equation in  $r_1, \dots, r_n$  and  $r$ . This rational equation can then be transformed into a polynomial equation  $g = 0$  where  $g \in \mathbb{Q}[r_1, \dots, r_k]$  as in (3). That the polynomial will be  $D_n$ -symmetric is clear from geometry: it

does not matter which angle we label  $\theta_1$  (rotation) or whether we do our numbering clockwise or counter-clockwise (reflection.)

EXAMPLE: For  $n = 3$  we have the equation  $f = C_3 = (x_1^2 + x_2^2 + x_3^2 - 2x_1x_2x_3 - 1)^2 = 0$  and by substitution of the radii into the equation obtain

$$\begin{aligned}
C_3 &= \left( \left( \frac{(r+r_1)^2 + (r+r_2)^2 - (r_1+r_2)^2}{2(r+r_1)(r+r_2)} \right)^2 + \left( \frac{(r+r_2)^2 + (r+r_3)^2 - (r_2+r_3)^2}{2(r+r_2)(r+r_3)} \right)^2 \right. \\
&\quad \left. + \left( \frac{(r+r_3)^2 + (r+r_1)^2 - (r_3+r_1)^2}{2(r+r_3)(r+r_1)} \right)^2 - 2 \left( \frac{(r+r_1)^2 + (r+r_2)^2 - (r_1+r_2)^2}{2(r+r_1)(r+r_2)} \right) \right. \\
&\quad \left( \frac{(r+r_2)^2 + (r+r_3)^2 - (r_2+r_3)^2}{2(r+r_2)(r+r_3)} \right) \left( \frac{(r+r_3)^2 + (r+r_1)^2 - (r_3+r_1)^2}{2(r+r_3)(r+r_1)} \right) - 1 \Big)^2 \\
&= \frac{16}{(r+r_1)^4(r+r_2)^4(r+r_3)^4} \left( -2r_1^2r_2r_3r^2 + r_1^2r_3^2r^2 + r_2^2r_3^2r^2 + r_1^2r_2^2r^2 - 2r_1r_2r_3^2r^2 \right. \\
&\quad \left. - 2r_1r_2^2r_3r^2 - 2r_1^2r_2^2r_3r - 2r_1r_2^2r_3^2r - 2r_1^2r_2r_3^2r + r_1^2r_2^2r_3^2 \right)^2 \\
&= 0.
\end{aligned}$$

Hence, our polynomial  $g$  is then given by

$$\begin{aligned}
g &= 16 \left( -2r_1^2r_2r_3r^2 + r_1^2r_3^2r^2 + r_2^2r_3^2r^2 + r_1^2r_2^2r^2 - 2r_1r_2r_3^2r^2 \right. \\
&\quad \left. - 2r_1r_2^2r_3r^2 - 2r_1^2r_2^2r_3r - 2r_1r_2^2r_3^2r - 2r_1^2r_2r_3^2r + r_1^2r_2^2r_3^2 \right)^2.
\end{aligned}$$

We now write

$$r^{12}g\left(\frac{r_1}{r}, \frac{r_2}{r}, \frac{r_3}{r}\right) = r^8g_8 + r^6g_6 + r^4g_4 - r^2g_2 + g_0 \in \mathbb{Q}[r_1, r_2, r_3][r],$$

where

$$\begin{aligned}
g_8 &= 16(r_2^4r_3^4 + r_1^4r_2^4 + r_1^4r_3^4 + 4r_1^3r_2^3r_3^2 - 4r_1^4r_2r_3^3 + 4r_1^2r_2^3r_3^3 - 4r_1^4r_2^3r_3 - 4r_1^3r_2^4r_3 \\
&\quad + 6r_1^2r_2^2r_3^4 - 4r_1^3r_2r_3^4 - 4r_1r_2^3r_3^4 - 4r_1r_2^4r_3^3 + 6r_1^4r_2^2r_3^2 + 6r_1^2r_2^4r_3^2 + 4r_1^3r_2^2r_3^3), \\
g_6 &= 64(r_1^3r_2^2r_3^4 - r_1^4r_2r_3^4 + 6r_1^3r_2^3r_3^3 + r_1^4r_2^2r_3^3 + r_1^4r_2^3r_3^2 + r_1^3r_2^4r_3^2 + r_1^2r_2^3r_3^4 + r_1^2r_2^4r_3^3 \\
&\quad - r_1r_2^4r_3^4 - r_1^4r_2^4r_3), \\
g_4 &= 32(3r_1^4r_2^2r_3^4 + 2r_1^4r_2^3r_3^3 + 2r_1^3r_2^3r_3^4 + 3r_1^4r_2^4r_3^2 + 3r_1^2r_2^4r_3^4 + 2r_1^3r_2^4r_3^3), \\
g_2 &= 64(r_1^3r_2^4r_3^4 - r_1^4r_2^3r_3^4 - r_1^4r_2^4r_3^3), \\
g_0 &= 16r_1^4r_2^4r_3^4,
\end{aligned}$$

and each of these  $g_i \in \mathbb{Q}[r_1, r_2, r_3]$  is a  $D_3$ -symmetric polynomial.

In general, for the terms with degree of  $\delta \in \{0, \dots, d\}$ , then  $r^d$  will cancel out all the denominators and we will be left with a term  $r^{d-\delta}g_{d-\delta}$  where  $g_{d-\delta}$  is an element of  $\mathbb{Q}[r_1, \dots, r_n]$ . That  $g_{d-\delta}$  will be  $D_n$ -symmetric follows from the  $D_n$ -symmetry of  $g$  and hence, viewing  $g$  as a polynomial in  $r$  alone, each coefficient for each power of  $r$  is also  $D_n$ -symmetric.

### 3 The polynomial of the $n$ -petal flower and its irreducibility

This section forms the main contribution and results of the paper. We will show that for each  $n \geq 2$  we have  $C_n = P_n^2$ , where  $P_n$  is an irreducible polynomial for  $n \geq 2$ , and  $P_n$  is symmetric for  $n \geq 3$ . To proceed we need some preliminary definitions and results.

**Definition 3.1** For  $n \in \mathbb{N}$  let  $G_n^*$  be the Galois group of automorphisms on  $\mathbb{Q}(x_1, \dots, x_n, y_1, \dots, y_n)$  that fixes the field  $\mathbb{Q}(x_1, \dots, x_n)$ . Also, let  $G_n$  be the Galois group of automorphisms on  $\mathbb{Q}(x_1, \dots, x_n, y_i y_j : i < j)$  that fixes the field  $\mathbb{Q}(x_1, \dots, x_n)$ . That is,

$$\begin{aligned} G_n^* &= \text{Gal}(\mathbb{Q}(x_1, x_2, \dots, x_n, y_1, \dots, y_n) / \mathbb{Q}(x_1, \dots, x_n)), \\ G_n &= \text{Gal}(\mathbb{Q}(x_1, x_2, \dots, x_n, y_i y_j : i < j) / \mathbb{Q}(x_1, \dots, x_n)), \end{aligned}$$

where  $x_1, \dots, x_n$  are algebraically independent indeterminates and  $y_i = \sqrt{1 - x_i^2}$  for each  $i$ , that is  $y_i$  is one root of  $X^2 + x_i^2 - 1 = 0 \in \mathbb{Q}(x_1, \dots, x_n)[X]$ .

**Lemma 3.2** For  $n \geq 1$  we have  $G_n^* \cong \mathbb{Z}_2^n$  and  $G_n \cong \mathbb{Z}_2^{n-1}$ .

*Proof.* For  $G_n^*$ , each  $y_i$  is the root of an irreducible quadratic polynomial  $X^2 - (1 - x_i^2)$  from the ring  $\mathbb{Q}(x_1, \dots, x_n, y_1, \dots, y_{i-1})[X]$ , which is the minimum polynomial of  $y_i$  for each  $i$ . Hence we have  $G_n^* \cong \mathbb{Z}_2^n$ .

For  $G_n$ , each  $y_i y_j$  with  $i < j$  is also the root of an irreducible quadratic polynomial  $X^2 - (1 - x_i^2)(1 - x_j^2) \in \mathbb{Q}(x_1, \dots, x_n)[X]$ . However, every element of  $\mathbb{Q}(x_1, x_2, \dots, x_n, y_i y_j : i < j)$  can be written as a rational function in terms of only elements of the form  $y_i y_{i+1}$  as follows:

$$y_i y_j = \frac{(y_i y_{i+1})(y_{i+1} y_{i+2}) \cdots (y_{j-1} y_j)}{y_{i+1}^2 \cdots y_{j-1}^2} = \frac{(y_i y_{i+1})(y_{i+1} y_{i+2}) \cdots (y_{j-1} y_j)}{(1 - x_{i+1}^2) \cdots (1 - x_{j-1}^2)}.$$

So we have that

$$\mathbb{Q}(x_1, x_2, \dots, x_n, y_i y_j : i < j) = \mathbb{Q}(x_1, x_2, \dots, x_n, y_i y_{i+1} : 1 \leq i < n).$$

Each term  $y_i y_{i+1}$  is a root of an irreducible quadratic polynomial  $X^2 - (1 - x_i^2)(1 - x_{i+1}^2)$  from the ring  $\mathbb{Q}(x_1, \dots, x_n, y_1 y_2, \dots, y_{i-1} y_i)[X]$ , which is the minimal polynomial of  $y_i y_{i+1}$  for each  $i \in \{1, \dots, n-1\}$ . Therefore we have that  $G_n \cong \mathbb{Z}_2^{n-1}$ .  $\square$

**Lemma 3.3** For  $n \in \mathbb{N}$ , the group  $G_n \cong \mathbb{Z}_2^{n-1}$  can be presented as

$$G_n = \langle \sigma_1, \dots, \sigma_{n-1} : \sigma_i^2 = e, \sigma_i \sigma_j = \sigma_j \sigma_i \rangle,$$

where each  $\sigma_i$  is an automorphism fixing  $\mathbb{Q}(x_1, \dots, x_n)$  and

$$\sigma_i(y_j y_{j+1}) = \begin{cases} -y_j y_{j+1} & \text{if } i = j \\ y_j y_{j+1} & \text{if } i \neq j. \end{cases}$$

*Proof.* Since  $(y_i y_{i+1})^2 = (1 - x_i^2)(1 - x_{i+1}^2)$  and the Galois group  $G_n$  is fixing the  $x_i$ , the only possible automorphisms are  $\sigma(y_i y_{i+1}) = -y_i y_{i+1}$  and  $\sigma(y_i y_{i+1}) = y_i y_{i+1}$ . We can then generate the group as in the statement of the theorem with  $n-1$  generators  $\sigma_i$ .  $\square$

**Corollary 3.4** For every  $\sigma \in G_n$ , let  $s_{\sigma;j} \in \{-1, 1\}$  be such that  $\sigma(y_j y_{j+1}) = s_{\sigma;j} y_j y_{j+1}$ . Then for every  $i < j$  we have  $\sigma(y_i y_j) = s_{\sigma;i} s_{\sigma;i+1} \cdots s_{\sigma;j} y_i y_j$ . In particular, if  $i < n$  then  $\sigma_{n-1}(y_i y_n) = -y_i y_n$  and if  $i > 1$  then  $\sigma_1(y_1 y_i) = -y_1 y_i$ .

We are now able to give a precise definition of  $C_n$  from Section 2 for each  $n \in \mathbb{N}$ .

**Definition 3.5** For  $n \in \mathbb{N}$ , define the polynomial  $C_n \in \mathbb{Q}[x_1, \dots, x_n]$  by

$$C_n(x_1, \dots, x_n) = \prod_{\sigma \in G_n} (\sigma(\text{EC}_n) - 1).$$

From Definition 3.5 we see that  $C_n$  is indeed symmetric in  $x_1, \dots, x_n$ .

EXAMPLE: For  $n = 2$  we have  $G_2 = \langle \sigma \rangle \cong \mathbb{Z}_2$  where  $\sigma(y_1 y_2) = -y_1 y_2$  and  $\sigma^2 = e$ , and hence

$$\prod_{\sigma \in G_2} (\sigma(\text{EC}_2) - 1) = (x_1 x_2 - y_1 y_2 - 1)(x_1 x_2 - \sigma(y_1 y_2) - 1) = (x_1 - x_2)^2 = C_2(x_1, x_2).$$

For  $n = 3$ , we have  $G_3 = \langle \sigma_1, \sigma_2 \rangle$  where  $\sigma_1(y_1 y_2) = -y_1 y_2$  and  $\sigma_2(y_2 y_3) = -y_2 y_3$  and hence

$$\begin{aligned} \prod_{\sigma \in G_3} (\sigma(\text{EC}_3) - 1) &= (x_1 x_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3 - y_1 y_2 x_3 - 1) \\ &\quad \cdot (x_1 x_2 x_3 - x_1 y_2 y_3 - \sigma_1(y_1 x_2 y_3) - \sigma_1(y_1 y_2 x_3) - 1) \\ &\quad \cdot (x_1 x_2 x_3 - \sigma_2(x_1 y_2 y_3) - \sigma_2(y_1 x_2 y_3) - y_1 y_2 x_3 - 1) \\ &\quad \cdot (x_1 x_2 x_3 - \sigma_1 \sigma_2(x_1 y_2 y_3) - \sigma_1 \sigma_2(y_1 x_2 y_3) - \sigma_1 \sigma_2(y_1 y_2 x_3) - 1) \\ &= (x_1 x_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3 - y_1 y_2 x_3 - 1) \\ &\quad \cdot (x_1 x_2 x_3 - x_1 y_2 y_3 + y_1 x_2 y_3 + y_1 y_2 x_3 - 1) \\ &\quad \cdot (x_1 x_2 x_3 + x_1 y_2 y_3 + y_1 x_2 y_3 - y_1 y_2 x_3 - 1) \\ &\quad \cdot (x_1 x_2 x_3 - x_1 y_2 y_3 + y_1 x_2 y_3 - y_1 y_2 x_3 - 1) \\ &= (x_1^2 + x_2^2 + x_3^2 - 2x_1 x_2 x_3 - 1)^2. \end{aligned}$$

By Lemma 2.1, each of the  $2^{n-2}$  terms of  $\text{ES}_{n-1}$  in terms of  $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}$  contains positive odd factors of  $y_i$  for  $i \leq n-1$ . Hence  $\sigma_{n-1} \in G_n$  fixes  $\mathbb{Q}(x_1, \dots, x_n, y_1 y_2, \dots, y_{n-2} y_{n-1})$  and  $\sigma_{n-1}(y_{n-1} y_n) = -y_{n-1} y_n$ . Noting this we then have by Corollary 3.4 the following:

**Claim 3.6** For  $n \geq 2$  we have  $G_n = G_{n-1} \cup G_{n-1} \sigma_{n-1} = G_{n-1} \cup \sigma_{n-1} G_{n-1}$  where  $\sigma_{n-1}(y_n \text{ES}_{n-1}) = -y_n \text{ES}_{n-1}$ .

If  $G_n$  is presented as in Lemma 3.3, then  $\sigma_{n-1} \in G_n$  fixes  $\mathbb{Q}(x_1, \dots, x_n, y_1 y_2, \dots, y_{n-2} y_{n-1})$  and  $\sigma_{n-1}(y_{n-1} y_n) = -y_{n-1} y_n$ .

**Lemma 3.7** For  $n \in \mathbb{N}$  let  $G_n$  be presented as in Lemma 3.3. Then

$$(\text{EC}_n - 1)(\sigma_{n-1}(\text{EC}_n) - 1) = (x_n - \text{EC}_{n-1})^2.$$

In particular,  $\text{EC}_n = 1$  implies  $x_n = \text{EC}_{n-1}$ .

*Proof.* By Claim 3.6 we have  $\sigma_{n-1}(y_n \text{ES}_{n-1}) = -y_n \text{ES}_{n-1}$  and hence

$$\begin{aligned}
(\text{EC}_n - 1)(\sigma_{n-1}(\text{EC}_n) - 1) &= (x_n \text{EC}_{n-1} - y_n \text{ES}_{n-1} - 1)(\sigma_{n-1}(x_n \text{EC}_{n-1} - y_n \text{ES}_{n-1}) - 1) \\
&= (x_n \text{EC}_{n-1} - y_n \text{ES}_{n-1} - 1)(x_n \text{EC}_{n-1} + y_n \text{ES}_{n-1} - 1) \\
&= (x_n \text{EC}_{n-1} - 1)^2 - y_n^2 \text{ES}_{n-1}^2 \\
&= (x_n \text{EC}_{n-1} - 1)^2 - (1 - x_n^2)(1 - \text{EC}_{n-1}^2) \\
&= (x_n - \text{EC}_{n-1})^2.
\end{aligned}$$

□

REMARK: Note that (2) implies directly that  $\cos(\theta_1 + \dots + \theta_h) = \cos(\theta_{h+1} + \dots + \theta_n)$  whenever  $h + k = n$ , and hence the equation  $\text{EC}_h(x_1, \dots, x_h) = \text{EC}_k(x_{h+1}, \dots, x_n)$  which itself implies  $x_n = \text{EC}_{n-1}$  by letting  $h = n - 1$  and  $k = 1$ .

**Corollary 3.8** *For  $n \geq 2$  we have  $C_n = P_n^2$  where*

$$P_n := \prod_{\sigma \in G_{n-1}} (x_n - \sigma(\text{EC}_{n-1})).$$

*Proof.* By Lemma 3.7 we obtain:

$$\begin{aligned}
C_n &= \prod_{\sigma \in G_n} (\sigma(\text{EC}_n) - 1) \\
&= \prod_{\sigma \in \sigma_n G_{n-1} \cup G_{n-1}} (\sigma(\text{EC}_n) - 1) \\
&= \prod_{\sigma \in G_{n-1}} (\sigma(\text{EC}_n) - 1)(\sigma_n \sigma(\text{EC}_n) - 1) \\
&= \prod_{\sigma \in G_{n-1}} \sigma((\text{EC}_n - 1)(\sigma_n(\text{EC}_n) - 1)) \\
&= \prod_{\sigma \in G_{n-1}} \sigma((x_n - \text{EC}_{n-1})^2) \\
&= \prod_{\sigma \in G_{n-1}} (x_n - \sigma(\text{EC}_{n-1}))^2 \\
&= P_n^2
\end{aligned}$$

where  $P_n = \prod_{\sigma \in G_{n-1}} (x_n - \sigma(\text{EC}_{n-1}))$ . □

By exactly the same token as Claim 3.6, Lemma 3.7, and Corollary 3.8, we obtain analogous results by reordering the variables  $y_1, \dots, y_n$  in the reverse order:  $y_n, y_{n-1}, \dots, y_1$ . Namely, if  $\sigma_i \in G_n$  is the field automorphism of  $\mathbb{Q}(x_1, \dots, x_n, y_1 y_2, y_2 y_3, \dots, y_{n-1} y_n)$  with  $\sigma_i(y_i y_{i+1}) = -y_i y_{i+1}$  fixing  $\mathbb{Q}(x_1, \dots, x_n)$  and each  $y_j y_{j+1}$  for  $j \neq i$  (as in Lemma 3.3) then we have the following:

**Claim 3.9** *If  $n \geq 2$  then  $G_n = G'_{n-1} \cup \sigma_1 G'_{n-1} = G'_{n-1} \cup G'_{n-1} \sigma_1$  where  $G'_{n-1} = \langle \sigma_2, \dots, \sigma_{n-1} \rangle$ , a subgroup of  $G_n = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$  and  $\sigma_1(y_1 \text{ES}_{n-1}(x_2, \dots, x_n)) = -y_1 \text{ES}_{n-1}(x_2, \dots, x_n)$ .*



*Proof.* By Lemma 2.1, each of the  $2^{n-2}$  terms of  $\text{ES}_{n-1}(x_2, \dots, x_n) = \text{ES}_{n-1}(x_2, \dots, x_n, y_2, \dots, y_n)$  (by substituting  $y_i = \sqrt{1 - x_i^2}$  for each  $i = 2, \dots, n$ ) has positive odd factors of  $y_i$  for  $i \geq 2$ . Hence the claim follows by Corollary 3.4.  $\square$

Similarly to Lemma 3.7 we now have the following.

**Lemma 3.10** *If  $\sigma_1 \in G_n$  is as above then*

$$(\text{EC}_n - 1)(\sigma_1(\text{EC}_n - 1)) = (x_1 - \text{EC}_{n-1}(x_2, \dots, x_n))^2.$$

*Proof.* By Claim 3.9 we obtain

$$\begin{aligned} (\text{EC}_n - 1)(\sigma_1(\text{EC}_n - 1)) &= (x_1 \text{EC}_{n-1}(x_2, \dots, x_n) - y_1 \text{ES}_{n-1}(x_2, \dots, x_n) - 1) \\ &\quad \cdot (\sigma_1(x_1 \text{EC}_{n-1}(x_2, \dots, x_n) - y_1 \text{ES}_{n-1}(x_2, \dots, x_n)) - 1) \\ &= (x_1 \text{EC}_{n-1}(\widehat{x}_1) - y_1 \text{ES}_{n-1}(\widehat{x}_1) - 1) \\ &\quad \cdot (x_1 \text{EC}_{n-1}(\widehat{x}_1) + y_1 \text{ES}_{n-1}(\widehat{x}_1) - 1) \\ &= (x_1 \text{EC}_{n-1}(\widehat{x}_1) - 1)^2 - y_1^2 \text{ES}_{n-1}(\widehat{x}_1)^2 \\ &= (x_1 \text{EC}_{n-1}(\widehat{x}_1) - 1)^2 - (1 - x_1^2)(1 - \text{EC}_{n-1}(\widehat{x}_1))^2 \\ &= (x_1 - \text{EC}_{n-1}(\widehat{x}_1))^2, \end{aligned}$$

where  $(\widehat{x}_1) = (x_2, \dots, x_n)$  as above.  $\square$

**Corollary 3.11** *For  $n \geq 3$  we have*

$$C_n = \prod_{\sigma \in G'_{n-1}} (x_1 - \sigma(\text{EC}_{n-1}(\widehat{x}_1)))^2.$$

*Proof.* By Lemma 3.10 we obtain as in the proof of Corollary 3.8

$$\begin{aligned} C_n &= \prod_{\sigma \in G_n} (\sigma(\text{EC}_n) - 1) \\ &= \prod_{\sigma \in G'_{n-1} \cup \sigma_1 G'_{n-1}} (\sigma(\text{EC}_n) - 1) \\ &= \prod_{\sigma \in G_{n-1}} (\sigma(\text{EC}_n) - 1)(\sigma \sigma_1(\text{EC}_n) - 1) \\ &= \prod_{\sigma \in G_{n-1}} \sigma(((\text{EC}_n) - 1)(\sigma_1(\text{EC}_n) - 1)) \\ &= \prod_{\sigma \in G_{n-1}} \sigma((x_1 - \text{EC}_{n-1}(\widehat{x}_1))^2) \\ &= \prod_{\sigma \in G'_{n-1}} (x_1 - \sigma(\text{EC}_{n-1}(\widehat{x}_1)))^2. \end{aligned}$$

$\square$

REMARK: For  $n = 1$  we have  $P_1 = C_1 = x_1 - 1$ . For  $n = 2$  we have (as defined in Corollary 3.8)  $P_2 = x_2 - x_1$ . However, this is a matter of taste, since we could have set  $P_2 = x_1 - x_2$ . The case  $n = 2$  is the only one where  $C_2(x_1, x_2)$  is symmetric while  $P_2$  is not.

By Corollary 3.11 we obtain  $C_n = Q_n^2$  where

$$Q_n = \prod_{\sigma \in G_{n-1}} (x_1 - \sigma(\text{EC}_{n-1}(\hat{x}_1))).$$

Since  $P_n^2 = C_n = Q_n^2$ , then as elements in a polynomial ring over a field, an integral domain, we get  $0 = P_n^2 - Q_n^2 = (P_n - Q_n)(P_n + Q_n)$  and hence for each  $n \geq 2$  we have  $Q_n = P_n$  or  $Q_n = -P_n$ .

For  $n = 2$  we obtain  $P_2 = x_2 - x_1$  and  $Q_2 = x_1 - x_2$  so  $Q_2 = -P_2$ .

For  $n \geq 3$  we first note that by evaluating  $\text{EC}_{n-1}(\hat{x}_n)$  and  $\text{EC}_{n-1}(\hat{x}_1)$  at  $x_2 = \dots = x_{n-1} = 1$  yields  $\text{EC}_{n-1}(\hat{x}_n)|_{x_2=\dots=x_{n-1}=1} = x_1$  and  $\text{EC}_{n-1}(\hat{x}_1)|_{x_2=\dots=x_{n-1}=1} = x_n$  and hence we obtain

$$\begin{aligned} P_n(x_1, 1, \dots, 1, x_n) &= \prod_{\sigma \in G_{n-1}} (x_n - x_1) = (x_n - x_1)^{2^{n-2}} \\ Q_n(x_1, 1, \dots, 1, x_n) &= \prod_{\sigma \in G_{n-1}} (x_1 - x_n) = (x_1 - x_n)^{2^{n-2}}. \end{aligned}$$

As  $n \geq 3$ , we have  $2^{n-2}$  is even and so  $(x_n - x_1)^{2^{n-2}} = (x_1 - x_n)^{2^{n-2}}$  and hence  $P_n(x_1, 1, \dots, 1, x_n) = Q_n(x_1, 1, \dots, 1, x_n)$ . Therefore we obtain the following:

**Corollary 3.12** *For  $n \geq 3$  we have  $Q_n = P_n$  and hence*

$$P_n = \prod_{\sigma \in G_{n-1}} (x_1 - \sigma(\text{EC}_{n-1}(\hat{x}_1))).$$

We now want to show that for  $n \geq 3$  the polynomial  $P_n$  is symmetric. Let  $n \geq 3$ . If  $\pi \in S_n$  is a permutation on  $\{1, \dots, n\}$  then  $\pi$  acts naturally on  $(x_1, \dots, x_n)$  by  $\pi(x_1, \dots, x_n) = (x_{\pi(1)}, \dots, x_{\pi(n)})$ . By definition of  $P_n$  in Corollary 3.8 we have

$$(P_n \circ \pi)(x_1, \dots, x_n) = P_n(x_{\pi(1)}, \dots, x_{\pi(n)}) = P_n(x_1, \dots, x_n)$$

or  $P_n \circ \pi = P_n \pi = P_n$  for all  $\pi \in S_n$  with  $\pi(n) = n$ . Likewise by Corollary 3.12 we have  $P_n \pi = P_n$  for all  $\pi \in S_n$  with  $\pi(1) = 1$ .

Let  $\tau \in S_n$  be an arbitrary transposition  $\tau = (i, j)$ . If  $\{i, j\} \subseteq \{1, \dots, n-1\}$  or  $\{i, j\} \subseteq \{2, \dots, n\}$  then by the above,  $P_n \tau = P_n$ . Otherwise if  $\tau = (1, n)$  then since  $n \geq 3$  there is an  $l \in \{2, \dots, n-1\}$  such that we can write  $\tau = (1, n) = (1, l)(l, n)(1, l)$  where  $\{1, l\} \subseteq \{2, \dots, n\}$ . From the above, we therefore have

$$P_n \tau = P_n(1, n) = P_n(1, l)(l, n)(1, l) = P_n(l, n)(1, l) = P_n(1, l) = P_n.$$

Since each permutation  $\pi \in S_n$  is a composition of transpositions then we have  $P_n \pi = P_n$  for each  $\pi \in S_n$ .

**Theorem 3.13** *For  $n \geq 3$  the polynomial  $P_n = P_n(x_1, \dots, x_n)$  is symmetric.*

**Corollary 3.14** For  $n \geq 3$  and any  $i \in \{1, \dots, n\}$  we have

$$P_n = \prod_{\sigma \in G_{n-1}} \sigma(x_i - \text{EC}_{n-1}(\hat{x}_i)).$$

In particular, as a polynomial in  $x_i$ , then  $P_n$  is monic of degree  $2^{n-2}$  in each  $x_i$ .

By Corollary 3.14 and definition of  $C_{n-1}$  we obtain by letting  $x_i = 1$  the following:

**Observation 3.15** For  $n \geq 3$  then

$$P_n(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) = \prod_{\sigma \in G_{n-1}} (1 - \sigma(\text{EC}_{n-1}(\hat{x}_i))) = C_{n-1}(\hat{x}_i) = P_{n-1}(\hat{x}_i)^2.$$

Other more general equations and formulae hold as well. Let  $n \in \mathbb{N}$  and  $n_1 + \dots + n_k = n$ . If  $\sum_{i=1}^n \theta_i = 2\pi$  and  $x_i = \cos \theta_i$  for each  $i \in \{1, \dots, n\}$ , then for each  $l \in \{1, \dots, k\}$  let  $\phi_l = \theta_{n_1+\dots+n_{l-1}+1} + \dots + \theta_{n_1+\dots+n_l}$ . Then  $\sum_{l=1}^k \phi_l = 2\pi$  and hence if  $t_l = \cos(\phi_l)$  then by Corollary 3.14 we get

$$0 = P_k(t_1, \dots, t_l) = P_k(\text{EC}_{n_1}, \dots, \text{EC}_{n_k}),$$

where for each  $l \in \{1, \dots, k\}$  we have  $\text{EC}_{n_l} = \text{EC}_{n_l}(x_{n_1+\dots+n_{l-1}+1}, \dots, x_{n_1+\dots+n_l})$ . In particular for  $k = n-1$  and  $n_1 = \dots = n_{n-2} = 1$  and  $n_{n-1} = 2$ , we have  $P_{n-1}(x_1, \dots, x_{n-2}, \text{EC}_2(x_{n-1}, x_n)) = 0$ , something we can use to compute  $P_n$  recursively. Let  $\overline{\text{EC}_2}(x_j, x_{j+1}) = x_j x_{j+1} + y_j y_{j+1}$  be the conjugate of  $\text{EC}_2(x_j, x_{j+1})$ . Recall that by Claim 3.6 we have for  $n-1$  that

$$G_{n-1} = G_{n-2} \cup \sigma_{n-1} G_{n-2} = G_{n-2} \cup G_{n-2} \sigma_{n-1}$$

and  $\sigma_{n-2}(y_{n-1} \text{ES}_{n-2}) = -y_{n-1} \text{ES}_{n-2}$ .

**Lemma 3.16** For  $n \geq 3$  we have

$$(x_n - \text{EC}_{n-1})(x_n - \sigma_{n-2}(\text{EC}_{n-1})) = x_{n-1}^2 + x_n^2 - 1 - 2x_{n-1}x_n \text{EC}_{n-2}^2 + \text{EC}_{n-2}^2.$$

*Proof.* Since  $\text{EC}_{n-1} = x_{n-1} \text{EC}_{n-2} - y_{n-1} \text{ES}_{n-2}$ , we obtain by above

$$\begin{aligned} (x_n - \text{EC}_{n-1})(x_n - \sigma_{n-2}(\text{EC}_{n-1})) &= (x_n - x_{n-1} \text{EC}_{n-2} + y_{n-1} \text{ES}_{n-2}) \\ &\quad \cdot (x_n - x_{n-1} \text{EC}_{n-2} - y_{n-1} \text{ES}_{n-2}) \\ &= (x_n - x_{n-1} \text{EC}_{n-2})^2 - y_{n-1}^2 \text{ES}_{n-2}^2 \\ &= (x_n - x_{n-1} \text{EC}_{n-2})^2 - (1 - x_{n-1}^2)(1 - \text{EC}_{n-2}^2) \\ &= x_{n-1}^2 + x_n^2 - 1 - 2x_{n-1}x_n \text{EC}_{n-2}^2 + \text{EC}_{n-2}^2. \end{aligned}$$

□

By direct computation and the definition of  $P_{n-1}$ , since  $\text{EC}_2(x_i, x_{i+1}) = x_i x_{i+1} - y_i y_{i+1}$ , we get

$$\begin{aligned}
& P_{n-1}(x_1, \dots, x_{n-2}, \text{EC}_2(x_{n-1}x_n))P_{n-1}(x_1, \dots, x_{n-2}, \overline{\text{EC}_2}(x_{n-1}x_n)) \\
&= \prod_{\sigma \in G_{n-2}} (\text{EC}_2(x_{n-1}, x_n) - \sigma(\text{EC}_{n-2})) \prod_{\sigma \in G_{n-2}} (\overline{\text{EC}_2}(x_{n-1}, x_n) - \sigma(\text{EC}_{n-2})) \\
&= \prod_{\sigma \in G_{n-2}} (x_{n-1}x_n - y_{n-1}y_n - \sigma(\text{EC}_{n-2})) \prod_{\sigma \in G_{n-2}} (x_{n-1}x_n + y_{n-1}y_n - \sigma(\text{EC}_{n-2})) \\
&= \prod_{\sigma \in G_{n-2}} ((x_{n-1}x_n - \sigma(\text{EC}_{n-2}))^2 - y_{n-1}^2 y_n^2) \\
&= \prod_{\sigma \in G_{n-2}} ((x_{n-1}x_n - \sigma(\text{EC}_{n-2}))^2 - (1 - x_{n-1}^2)(1 - x_n^2)) \\
&= \prod_{\sigma \in G_{n-2}} (x_{n-1}^2 + x_n^2 - 1 - 2x_{n-1}x_n\sigma(\text{EC}_{n-2}^2) + \sigma(\text{EC}_{n-2}^2)) \\
&= \prod_{\sigma \in G_{n-2}} \sigma(x_{n-1}^2 + x_n^2 - 1 - 2x_{n-1}x_n\text{EC}_{n-2}^2 + \text{EC}_{n-2}^2).
\end{aligned}$$

From this we can prove the following:

**Theorem 3.17** *The polynomials  $P_n$  are completely determined by the following recursion:  $P_1 = x_1 - 1$ ,  $P_2 = x_2 - x_1$  and for  $n \geq 3$*

$$P_n = P_{n-1}(x_1, \dots, x_{n-2}, \text{EC}_2(x_{n-1}, x_n))P_{n-1}(x_1, \dots, x_{n-2}, \overline{\text{EC}_2}(x_{n-1}, x_n)).$$

*Proof.* By Lemma 3.16 and the preceding paragraph we get

$$\begin{aligned}
P_n &= \prod_{\sigma \in G_{n-1}} (x_n - \sigma(\text{EC}_{n-1})) \\
&= \prod_{\sigma \in G_{n-2} \cup \sigma_{n-1}G_{n-2}} (x_n - \sigma(\text{EC}_{n-1})) \\
&= \prod_{\sigma \in G_{n-2}} (x_n - \sigma(\text{EC}_{n-1}))(x_n - \sigma_{n-1}(\text{EC}_{n-1})) \\
&= \prod_{\sigma \in G_{n-2}} \sigma((x_n - (\text{EC}_{n-1}))(x_n - \sigma_{n-1}(\text{EC}_{n-1}))) \\
&= \prod_{\sigma \in G_{n-2}} \sigma(x_{n-1}^2 + x_n^2 - 1 - 2x_{n-1}x_n\text{EC}_{n-2}^2 + \text{EC}_{n-2}^2) \\
&= P_{n-1}(x_1, \dots, x_{n-2}, \text{EC}_2(x_{n-1}, x_n)) \cdot P_{n-1}(x_1, \dots, x_{n-2}, \overline{\text{EC}_2}(x_{n-1}, x_n)).
\end{aligned}$$

□

EXAMPLE: With the help of MAPLE [8] the first 5 polynomials  $P_n$  can now be computed quickly and efficiently by the recursion in Theorem 3.17.

$$\begin{aligned}
P_1 &= x_1 - 1. \\
P_2 &= x_2 - x_1. \\
P_3 &= P_2(x_1, \text{EC}_2(x_2, x_3))P_2(x_1, \overline{\text{EC}_2}(x_2, x_3)) \\
&= (x_2x_3 - y_2y_3 - x_1)(x_2x_3 + y_2y_3 - x_1) \\
&= x_1^2 + x_2^2 + x_3^2 - 2x_1x_2x_3 - 1. \\
P_4 &= P_3(x_1, x_2, \text{EC}_2(x_3, x_4))P_3(x_1, x_2, \overline{\text{EC}_2}(x_3, x_4)) \\
&= (x_1^2 + x_2^2 + (x_3x_4 - y_3y_4)^2 - 2x_1x_2(x_3x_4 - y_3y_4) - 1) \\
&\quad \cdot (x_1^2 + x_2^2 + (x_3x_4 + y_3y_4)^2 - 2x_1x_2(x_3x_4 + y_3y_4) - 1) \\
&= x_1^4 + x_2^4 + x_3^4 + x_4^4 - 2(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_1^2x_4^2 + x_1^2x_3^2 + x_2^2x_4^2) \\
&\quad + 4(x_1^2x_2^2x_3^2 + x_2^2x_3^2x_4^2 + x_1^2x_3^2x_4^2 + x_1^2x_2^2x_4^2) \\
&\quad + 4x_1x_2x_3x_4(2 - x_1^2 - x_2^2 - x_3^2 - x_4^2). \\
P_5 &= \text{a display of terms on two letter size pages, see Appendix B.}
\end{aligned}$$

The recursion given in Theorem 3.17, although fundamental for computation, is a special case of a more general recursion that  $P_n$  satisfies:

**Claim 3.18** *Let  $n, k \geq 2$  and  $n_1 + \dots + n_k = n$ . By the right interpretation of  $\sigma_i$  for each  $i \in [k]$  (and with some abuse of notation) then  $P_n = P_n(x_1, \dots, x_n)$  satisfies the following general recursion*

$$\begin{aligned}
P_n(x_1, \dots, x_n) &= \prod_{\substack{\sigma_i \in G_{n_i-1} \\ i \in [k]}} P_k(\sigma_1(\text{EC}_{n_1}(x_1, \dots, x_{n_1})), \sigma_2(\text{EC}_{n_2}(x_{n_1+1}, \dots, x_{n_1+n_2})), \\
&\quad \dots, \sigma_k(\text{EC}_{n_k}(x_{n-n_k+1}, \dots, x_n))).
\end{aligned}$$

As this more general recursion of Claim 3.18 will not be used to obtain our main result Theorem 3.20 here below, its proof in detail will be omitted. However, this can be proved using induction in stages using Theorem 3.17 as a stepping stone.

EXAMPLE: We demonstrate how Claim 3.18 works by using it to compute  $P_5$ , since  $n = 5$  is the smallest nontrivial example (with  $k \geq 3$ ) that can be generated using a recurrence from Claim 3.18

that is not an example of the special recurrence from Theorem 3.17:

$$\begin{aligned}
P_5(x_1, \dots, x_5) &= \prod_{\substack{\sigma'_1 \in G_1 = \langle \sigma_1 \rangle \\ \sigma'_2 \in G_2 = \{e\} \\ \sigma'_3 \in G_3 = \langle \sigma_3 \rangle}} P_3(\sigma'_1(\text{EC}_2(x_1, x_2)), \sigma'_2(\text{EC}_1(x_3)), \sigma'_3(\text{EC}_2(x_4, x_5))) \\
&= \prod_{\substack{\sigma'_1 \in G_1 = \langle \sigma_1 \rangle \\ \sigma'_2 \in G_2 = \{e\} \\ \sigma'_3 \in G_3 = \langle \sigma_3 \rangle}} P_3(\sigma'_1(x_1x_2 - y_1y_2), \sigma'_2(x_3), \sigma'_3(x_4x_5 - y_4y_5)) \\
&= P_3(x_1x_2 - y_1y_2, x_3, x_4x_5 - y_4y_5) \cdot P_3(x_1x_2 + y_1y_2, x_3, x_4x_5 - y_4y_5) \\
&\quad \cdot P_3(x_1x_2 - y_1y_2, x_3, x_4x_5 + y_4y_5) \cdot P_3(x_1x_2 + y_1y_2, x_3, x_4x_5 + y_4y_5) \\
&= ((x_1x_2 - y_1y_2)^2 + x_3^2 + (x_4x_5 - y_4y_5)^2 - 2(x_1x_2 - y_1y_2)x_3(x_4x_5 - y_4y_5) - 1) \\
&\quad \cdot ((x_1x_2 + y_1y_2)^2 + x_3^2 + (x_4x_5 - y_4y_5)^2 - 2(x_1x_2 + y_1y_2)x_3(x_4x_5 - y_4y_5) - 1) \\
&\quad \cdot ((x_1x_2 - y_1y_2)^2 + x_3^2 + (x_4x_5 + y_4y_5)^2 - 2(x_1x_2 - y_1y_2)x_3(x_4x_5 + y_4y_5) - 1) \\
&\quad \cdot ((x_1x_2 + y_1y_2)^2 + x_3^2 + (x_4x_5 + y_4y_5)^2 - 2(x_1x_2 + y_1y_2)x_3(x_4x_5 + y_4y_5) - 1).
\end{aligned}$$

Expanded, this last product yields the same expression for  $P_5$  as given in Appendix B.

Our final goal in this section, and our main result of the paper, is to prove the irreducibility of  $P_n$ . To illuminate our approach we state and prove the following simplest case, that  $P_3 = P_3(x_1, x_2, x_3)$  is irreducible.

Suppose  $P_3 = fg$  with  $f, g \in \mathbb{Q}[x_1, x_2, x_3]$ . Since  $P_3$  is monic in  $x_3$ , both  $f$  and  $g$  contain the variable  $x_3$ , and hence both  $f$  and  $g$  are of degree 1 in  $x_3$  (unless  $f$  or  $g = P_3$ .) Since  $P_3$  factors in  $\mathbb{Q}(x_1, x_2, y_1y_2)[x_3]$  as  $P_3 = (x_3 - x_1x_2 - y_1y_2)(x_3 - x_1x_2 + y_1y_2)$  by definition of  $P_3$ , then since  $\mathbb{Q}(x_1, x_2, y_1y_2)[x_3]$  is a UFD we must have

$$\{f, g\} = \{x_3 - x_1x_2 - y_1y_2, x_3 - x_1x_2 + y_1y_2\}$$

which contradicts the assumption that  $f, g \in \mathbb{Q}[x_1, x_2, x_3]$ . Hence we have the following observation:

**Observation 3.19** *The polynomial  $P_3(x_1, x_2, x_3)$  is irreducible over  $\mathbb{Q}$ .*

Note that the same argument holds if  $\mathbb{Q}$  is replaced with the complex field  $\mathbb{C}$  in the above.

We now use this same approach to prove the following:

**Theorem 3.20** *For each  $n \geq 3$  the polynomial  $P_n(x_1, \dots, x_n)$  is irreducible over  $\mathbb{Q}$ .*

We will prove Theorem 3.20 by induction on  $n$ , assuming that  $P_{n-1}$  is irreducible over  $\mathbb{Q}$ . But before we can delve into that, we need to prove the following:

**Lemma 3.21** *Let  $n \geq 3$ . If  $P_{n-1}$  is irreducible over  $\mathbb{Q}$  then  $P_{n-1}(\text{EC}_2(x_1, x_2), x_3, \dots, x_n) = P_{n-1}(x_1x_2 - y_1y_2, x_3, \dots, x_n)$  and  $P_{n-1}(\overline{\text{EC}_2}(x_1, x_2), x_3, \dots, x_n) = P_{n-1}(x_1x_2 + y_1y_2, x_3, \dots, x_n)$  are irreducible in  $\mathbb{Q}(x_1, x_2, y_1y_2)[x_3, \dots, x_n]$ .*

*Proof.* Let  $P_{n-1}^* := P_{n-1}(x_1x_2 - y_1y_2, x_3, \dots, x_n)$  and assume it factors as  $P_{n-1}^* = h^*k^*$  in the ring  $\mathbb{Q}(x_1, x_2, y_1y_2)[x_3, \dots, x_n]$ , where both  $h^*$  and  $k^*$  involve  $x_n$ . Since  $P_{n-1} = \prod_{\sigma \in G_{n-2}} (x_n - \sigma(\text{EC}_{n-2}))$ , we see that

$$P_{n-1}^* = \prod_{\sigma \in G_{n-2}} (x_n - \sigma(\text{EC}_{n-2}(x_1x_2 - y_1y_2, x_3, \dots, x_n))),$$

and hence both  $h^*$  and  $k^*$  must be products of these linear factors. In particular, we can evaluate  $P_{n-1}^* = h^*k^*$  at  $x_1 = 1$  and obtain

$$P_{n-1}(x_2, \dots, x_n) = (P_{n-1}^*)|_{x_1=1} = (h^*|_{x_1=1})(k^*|_{x_1=1}) = hk$$

in  $\mathbb{Q}(x_2)[x_3, \dots, x_n]$ , which is a UFD. By assumption  $P_{n-1}(x_2, \dots, x_n)$  is irreducible in the ring  $\mathbb{Q}[x_2, \dots, x_n] = \mathbb{Q}[x_2][x_3, \dots, x_n]$  and hence also in  $\mathbb{Q}(x_2)[x_3, \dots, x_n]$  (as a monic polynomial in  $x_n$ ). Therefore either  $h$  or  $k$  equals  $P_{n-1}(x_2, \dots, x_n)$ , which contradicts the fact that both  $h^*$  and  $k^*$  involve  $x_n$ . Hence  $P_{n-1}^*$  is irreducible. In the same way we obtain that  $P_{n-1}(x_1x_2 + y_1y_2, x_3, \dots, x_n)$  is irreducible.  $\square$

*Proof.* [Theorem 3.20] Let  $n \geq 3$  and assume that  $P_{n-1}$  is irreducible over  $\mathbb{Q}$ . Assume  $P_n = fg$  with  $f, g \in \mathbb{Q}[x_1, \dots, x_n]$ . We may assume  $f$  is irreducible. Let  $\phi_i : \mathbb{Q}[x_1, \dots, x_n] \rightarrow \mathbb{Q}[\hat{x}_i]$  be the evaluation at  $x_i = 1$ , that is  $\phi_i(F) = F(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ . Since  $\phi_i$  is a  $\mathbb{Q}$ -algebra homomorphism for each  $i \in [n]$  we have for  $i = 1$  that

$$\phi_1(P_n) = \phi_1(fg) = \phi_1(f)\phi_1(g) \in \mathbb{Q}[x_2, \dots, x_n].$$

But  $\phi_1(P_n) = P_{n-1}(x_2, \dots, x_n)^2 \in \mathbb{Q}[x_2, \dots, x_n]$ , which is a UFD. By the inductive hypothesis,  $P_{n-1}$  is irreducible in  $\mathbb{Q}[x_2, \dots, x_n]$ . Therefore,  $\phi_1(f) = P_{n-1} = \phi_1(g)$  (unless  $f = P_n$ , in which case we are done since  $f$  is irreducible).

Viewing  $f, g \in \mathbb{Q}[x_1, \dots, x_{n-1}][x_n]$ , then since  $P_n$  and  $P_{n-1}$  are monic in every variable  $x_i$  (and hence also in  $x_n$ ), we have

$$\deg_{x_n}(f) = \deg_{x_n}(g) = \frac{\deg_{x_n}(P_n)}{2} = 2^{n-3}.$$

By symmetry of  $P_n$  for  $n \geq 3$ , from Theorem 3.13 and Theorem 3.17 we have

$$P_n = P_{n-1}(\text{EC}_2(x_1, x_2), x_3, \dots, x_n)P_{n-1}(\overline{\text{EC}_2}(x_1, x_2), x_3, \dots, x_n)$$

in  $\mathbb{Q}(x_1, x_2, y_1y_2)[x_3, \dots, x_n]$ , which is a UFD. Since by assumption  $P_n = fg$  where  $f \in \mathbb{Q}[x_1, \dots, x_n]$  is irreducible and  $f|_{x_1=1} = \phi_1(f) = P_{n-1}(x_2, \dots, x_n)$ , which by assumption is irreducible. That  $f$  is also irreducible in  $\mathbb{Q}(x_1, x_2, y_1y_2)[x_3, \dots, x_n]$  can now be seen in the same way as in the proof of Lemma 3.21: namely, by evaluating at  $x_1 = 1$  and obtain a factorization of  $P_{n-1}(x_2, \dots, x_n)$ .

So we have

$$P_n = fg = P_{n-1}(\text{EC}_2, x_3, \dots, x_n)P_{n-1}(\overline{\text{EC}_2}, x_3, \dots, x_n)$$

in  $\mathbb{Q}(x_1, x_2, y_1y_2)[x_3, \dots, x_n]$ , which is a UFD. Therefore we have

$$f \in \{P_{n-1}(\text{EC}_2, x_3, \dots, x_n), P_{n-1}(\overline{\text{EC}_2}, x_3, \dots, x_n)\},$$

By repeated application of Observation 3.15 we obtain

$$P_{n-1}(\text{EC}_2, 1, \dots, 1) = P_1(\text{EC}_2)^{2^{n-2}} = (\text{EC}_2 - 1)^{2^{n-2}}$$

which is not contained in  $\mathbb{Q}[x_1, \dots, x_n]$ . Similarly  $P_{n-1}(\overline{\text{EC}_2}, x_3, \dots, x_n) \notin \mathbb{Q}[x_1, \dots, x_n]$  and hence we have a contradiction, since  $f \in \mathbb{Q}[x_1, \dots, x_n]$ .  $\square$

REMARK: Replacing  $\mathbb{Q}$  with  $\mathbb{C}$  in the previous proofs will yield the same result.

As a corollary we obtain the following, which is in fact equivalent to Theorem 3.20:

**Corollary 3.22** *For  $n \in \mathbb{N}$  we have  $[\mathbb{Q}(x_1, \dots, x_n, \text{EC}_n) : \mathbb{Q}(x_1, \dots, x_n)] = 2^{n-1}$ .  
In fact, for any  $m \leq n$  we have  $[\mathbb{Q}(x_1, \dots, x_n, \text{EC}_m) : \mathbb{Q}(x_1, \dots, x_n)] = 2^{m-1}$ .*

we conclude this section with a summarizing result:

**Corollary 3.23** *For  $1 \leq m \leq n$  we have*

- $\mathbb{Q}(x_1, \dots, x_n, \text{EC}_m) = \mathbb{Q}(x_1, \dots, x_n, y_1 y_2, \dots, y_{m-1} y_m)$
- $\text{Gal}(\mathbb{Q}(x_1, \dots, x_n, \text{EC}_m) / \mathbb{Q}(x_1, \dots, x_n))$   
 $= \text{Gal}(\mathbb{Q}(x_1, \dots, x_n, y_1 y_2, \dots, y_{m-1} y_m) / \mathbb{Q}(x_1, \dots, x_n))$   
 $\cong \mathbb{Z}_2^{m-1}$ .
- $P_{m+1}(x_1, \dots, x_m, X) \in \mathbb{Q}(x_1, \dots, x_m)[X]$  is the minimal polynomial of  $\text{EC}_m = \text{EC}_m(x_1, \dots, x_m)$  over  $\mathbb{Q}(x_1, \dots, x_m)$ .

## 4 Rational solutions for $n = 3$ and Descartes' circle theorem

Here we deal with the special case of  $n = 3$ , and we characterize all rational solutions for flowers with three petals, thereby obtaining all rational radii of four mutually tangent Soddy circles. We then compare our parametrization to an existing parametrization of the curvatures of four mutually tangent circles and show how our equation-free parameterization is an improvement on the existing one.

In general, to find all integer-radii coins forming an  $n$ -petal flower in the Euclidean plane, it is equivalent by scaling, to find all rational radii coins where the center coin is assumed to have radius one. Note that if the lengths of the sides of a triangle are rational, then the cosines of all its angles will be rational. The converse is not necessarily true, however. We will solve  $P_3 = 0$  over the rationals and use that to find rational radii that create a 3-petal flower, that is, rational Soddy circles. For a necessary first step, we will determine what the cosines must be.

### 4.1 Rational solutions of $P_3 = 0$ and rational Soddy circles

First, we have the irreducible polynomial  $P_3 = x_1^2 + x_2^2 + x_3^2 - 2x_1 x_2 x_3 - 1$ . We can solve for any one of the variables, say  $x_3$ , by definition of  $P_3$  and Lemma 3.7:

$$x_3 = \text{EC}_2(x_1, x_2) = x_1 x_2 - \sqrt{(1 - x_1^2)(1 - x_2^2)}. \quad (4)$$

For rational  $x_1$  and  $x_2$  it is clear that  $x_3$  will be rational if and only if the term under the radical is the square of a rational number. For  $i = 1, 2$  let  $x_i = \frac{p_i}{q_i}$  for with  $p_i, q_i \in \mathbb{Z}$ . By (4) we then obtain

$$x_3 = x_1 x_2 - \frac{1}{q_1 q_2} \sqrt{(q_1^2 - p_1^2)(q_2^2 - p_2^2)}.$$

Here  $(q_1^2 - p_1^2)(q_2^2 - p_2^2)$  is a square if and only if  $q_i^2 - p_i^2 = s_i^2 \beta$  for  $i = 1, 2$  where  $\beta$  is square-free integer. Here we need the following result in elementary number theory:



**Theorem 4.1** *Let  $\beta$  be a square-free integer. The integers  $x, y, z$  form a primitive solution to the Diophantine equation  $x^2 + \beta y^2 = z^2$  if and only if there are positive integers  $m$  and  $n$  and a factorization  $\beta = bc$  where  $bm^2$  and  $cn^2$  are relatively prime such that  $x = \frac{bm^2 - cn^2}{2}$ ,  $y = mn$ ,  $z = \frac{bm^2 + cn^2}{2}$ , where both  $m$  and  $n$  are odd or both are even, or  $x = bm^2 - cn^2$ ,  $y = 2mn$ ,  $z = bm^2 + cn^2$  otherwise.*

For a proof of Theorem 4.1, see Appendix A.

Since  $x_i = \frac{p_i}{q_i}$  for  $i = 1, 2$  we have by Theorem 4.1 that

$$x_1 = \frac{b_1 m_1^2 - c_1 n_1^2}{b_1 m_1^2 + c_1 n_1^2}, \quad x_2 = \frac{b_2 m_2^2 - c_2 n_2^2}{b_2 m_2^2 + c_2 n_2^2}, \quad (5)$$

where  $\beta = b_1 c_1 = b_2 c_2$  are two (not necessarily distinct) factorizations of the square-free integer  $\beta$ , and where  $m_i, n_i$  can be chosen from the nonnegative integers.

Suppose we have a 3-petal flower whose internal angles are  $\theta_1, \theta_2, \theta_3$  and their cosines are  $x_1, x_2, x_3$  respectively. By scaling, we assume the radius of the center coin to be one and the other three outer radii  $r_1, r_2$  and  $r_3$ . By the law of cosines, we obtain

$$x_1 = \frac{r_1 + r_2 - r_1 r_2 + 1}{r_1 + r_2 + r_1 r_2 + 1}, \quad x_2 = \frac{r_2 + r_3 - r_2 r_3 + 1}{r_2 + r_3 + r_2 r_3 + 1}, \quad x_3 = \frac{r_3 + r_1 - r_3 r_1 + 1}{r_3 + r_1 + r_3 r_1 + 1}.$$

Rewriting each equation for  $x_i$  as a polynomial equation in terms of  $r_i$  and  $r_{i+1}$  (where  $4 \equiv 1$  modulo 3) and then factoring in terms of  $r_i$  and  $r_{i+1}$  we obtain

$$\begin{aligned} \left(r_1 + \frac{x_1 - 1}{x_1 + 1}\right) \left(r_2 + \frac{x_1 - 1}{x_1 + 1}\right) &= \frac{2(1 - x_1)}{(x_1 + 1)^2}, \\ \left(r_2 + \frac{x_2 - 1}{x_2 + 1}\right) \left(r_3 + \frac{x_2 - 1}{x_2 + 1}\right) &= \frac{2(1 - x_2)}{(x_2 + 1)^2}, \\ \left(r_3 + \frac{x_3 - 1}{x_3 + 1}\right) \left(r_1 + \frac{x_3 - 1}{x_3 + 1}\right) &= \frac{2(1 - x_3)}{(x_3 + 1)^2}. \end{aligned}$$

Now we can solve the first and third equations for  $r_2$  and  $r_3$  respectively in terms of  $r_1, x_1, x_3$ . Substituting these into the second equation, we can then solve that for  $r_1$  in terms of  $x_1, x_2, x_3$  obtaining

$$r_1 = \frac{-1 - x_1 x_3 + x_3 + x_1 \pm \sqrt{2(1 - x_1)(1 - x_2)(1 - x_3)}}{2x_2 - x_1 + x_1 x_3 - 1 - x_3}. \quad (6)$$

Putting  $x_1$  and  $x_2$  from (5) into (4) we obtain

$$x_3 = x_1 x_2 - \sqrt{(1 - x_1^2)(1 - x_2^2)} = \frac{(b_1 m_1^2 - c_1 n_1^2)(b_2 m_2^2 - c_2 n_2^2) - 4m_1 m_2 n_1 n_2 \beta}{(b_1 m_1^2 + c_1 n_1^2)(b_2 m_2^2 + c_2 n_2^2)}.$$

Substituting this expressions for  $x_3$  and those of  $x_1$  and  $x_2$  from (5) into (6), we get an expression for  $r_1$  in terms of  $b_1, b_2, c_1, c_2, m_1, m_2, n_1, n_2$ :

$$\begin{aligned} r_1 &= \frac{n_1(b_2 c_1^2 m_2^2 n_1^3 + 2\beta c_1 m_1 m_2 n_1^2 n_2 + b_1 c_1 c_2 m_1^2 n_1 n_2^2)}{b_1 c_1 c_2 m_1^2 n_1^2 n_2^2 - b_2 c_1^2 m_2^2 n_1^4 + c_1^2 c_2 n_1^4 n_2^2 - 2\beta c_1 m_1 m_2 n_1^3 n_2 + b_1^2 c_2 m_1^4 n_2^2} \\ &\pm \frac{n_1 n_2 (b_1 m_1^2 + c_1 n_1^2) \sqrt{c_1 c_2 (b_1 c_2 m_1^2 n_2^2 + 2\beta m_1 m_2 n_1 n_2 + b_2 c_1 m_2^2 n_1^2)}}{b_1 c_1 c_2 m_1^2 n_1^2 n_2^2 - b_2 c_1^2 m_2^2 n_1^4 + c_1^2 c_2 n_1^4 n_2^2 - 2\beta c_1 m_1 m_2 n_1^3 n_2 + b_1^2 c_2 m_1^4 n_2^2}. \end{aligned}$$

Using the fact that  $\beta = b_1c_1 = b_2c_2$ , the expression under the square root can be reduced to  $\beta(c_2m_1n_2 + c_1m_2n_1)^2$ . Thus this expression for  $r_1$  will only yield a perfect square when  $\beta = 1$ . Therefore,  $r_1$  is rational if and only if  $q_i^2 - p_i^2 = s_i^2$  for  $i = 1, 2$ , or in other words when  $1 - x_i^2$  is a perfect square for  $i = 1, 2$ . This means that both  $\cos \theta_i$  and  $\sin \theta_i$  are rational for  $i = 1, 2, 3$ .

**Proposition 4.2** *The 3-petaled flower with the center coin of radius one can have the outer coins of rational radii  $r_1, r_2, r_3$  if and only if the internal angles  $\theta_1, \theta_2, \theta_3$  have both rational cosines and sines for  $i = 1, 2, 3$ .*

Proposition 4.2 shows a property that is very special for the  $n$ -petal flower with rational radii when  $n = 3$ . We now can write a “nice” parametrization for the cosines  $x_i$  and the radii  $r_i$  in the case when  $n = 3$ . Let  $m_1, n_1, m_2, n_2 \in \mathbb{N}$ . Then

$$x_1 = \frac{m_1^2 - n_1^2}{m_1^2 + n_1^2}, \quad x_2 = \frac{m_2^2 - n_2^2}{m_2^2 + n_2^2}, \quad x_3 = \frac{(m_1^2 - n_1^2)(m_2^2 - n_2^2) - 4m_1m_2n_1n_2}{(m_1^2 + n_1^2)(m_2^2 + n_2^2)}. \quad (7)$$

Putting these into (6) and the similar equations for  $r_2$  and  $r_3$  we obtain the rational forms for  $r_1, r_2, r_3$  that contain all rational radii for the outer coins of a 3-petal flower with center coin of radius one:

$$\begin{aligned} r_1 &= \frac{n_1(m_1n_2 + m_2n_1)}{-m_1^2n_2 - n_1^2n_2 \pm (m_1n_1n_2 + m_2n_1^2)} \\ r_2 &= \frac{n_1n_2}{-n_1n_2 \pm (m_2n_1 + m_1n_2)} \\ r_3 &= \frac{n_2(m_1n_2 + m_2n_1)}{-m_1n_2^2 - m_2n_1n_2 \pm (n_1n_2^2 + m_2^2n_1)}. \end{aligned}$$

We will determine which range of the parameters will yield meaningful solutions in what follows as well as the signs of the terms in the denominator.

**Observation 4.3** *If  $\theta_1, \theta_2, \theta_3$  are the internal angles of a 3-petaled flower, then  $90^\circ < \theta_i < 180^\circ$  for each  $i = 1, 2, 3$  and these three inequalities are all sharp.*

*Proof.* As each  $\theta_i$  is an angle in a triangle formed by the mutually touching three coins, we have that  $\theta_i < 180^\circ$ . On the other hand, keeping the radius of the center coin fixed (say, at  $r = 1$ ) and letting  $r_i = r_{i+1} \rightarrow \infty$ , we see that  $\theta_i \rightarrow 180^\circ$  from below. We also see from this scenario that the other two angles tend to  $90^\circ$  from below.

What remains to show is that  $\theta_i > 90^\circ$  for each  $i$ . It suffices to show this for  $i = 1$ . By keeping the radii  $r_1$  and  $r_2$  fixed and letting  $r_3 \rightarrow \infty$ , the radius  $r$  of the central coin will increase and  $\theta_1$ , the angle between the first and second coins, will decrease. Figure 2 illustrates this situation. It suffices to show that  $\theta_1 > 90^\circ$  for this case. If we start with Figure 2 and draw a line parallel to the infinite circle that goes through the center of the central coin, we have 2 right triangles with side lengths  $r_i - r, r_i + r$  and, by the Pythagorean theorem,  $2\sqrt{r_i r}$  for each  $i = 1, 2$ . Therefore, the length of the segment forming the bottom of the rhombus, formed by the center of the two outer circles and their touching points to the infinite circle, is  $2(\sqrt{r_1 r} + \sqrt{r_2 r})$ . We can now draw a segment parallel to this segment and passing through the center of the coin with the smaller radius. Without loss of generality we may assume  $r_1 \leq r_2$ . Now we have a right triangle with

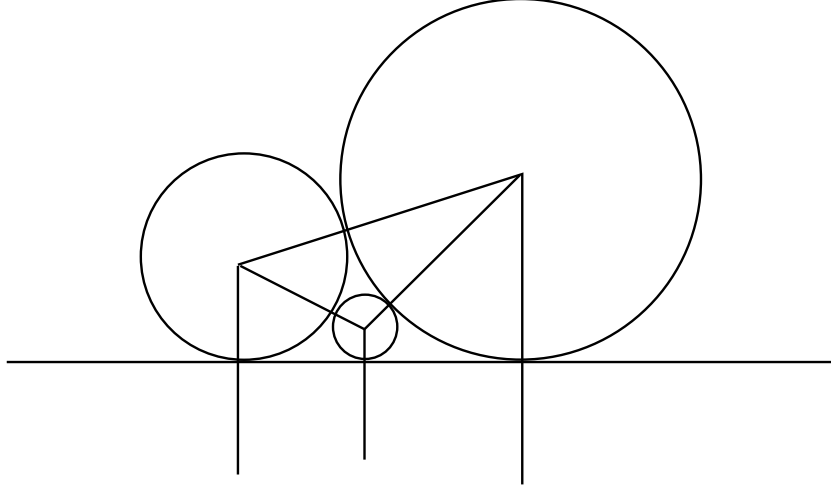


Figure 2: A 3-petaled flower where the radius of one petal is increased to infinity.

side lengths  $2(\sqrt{r_1 r} + \sqrt{r_2 r})$ ,  $r_2 - r_1$  and  $r_1 + r_2$  and hence by the Pythagorean theorem we have  $4(\sqrt{r_1 r} + \sqrt{r_2 r})^2 + (r_2 - r_1)^2 = (r_1 + r_2)^2$ , which can be solved for  $r$ , obtaining

$$r = \frac{r_1 r_2}{(\sqrt{r_1} + \sqrt{r_2})^2}.$$

With this expression for  $r$ , it suffices to show that  $(r_1 + r_2)^2 > (r + r_1)^2 + (r + r_2)^2$ , which implies  $\theta_1 > 90^\circ$ :

$$\begin{aligned} & (r_1 + r_2)^2 \\ &= \frac{(r_1 + r_2)^2 (\sqrt{r_1} + \sqrt{r_2})^4}{(\sqrt{r_1} + \sqrt{r_2})^4} \\ &= \frac{r_1^4 + r_2^4 + (14r_1^2 r_2^2 + 8r_1^3 r_2 + 12r_1^{5/2} r_2^{3/2} + 12r_1^{3/2} r_2^{5/2} + 8r_1 r_2^3) + 4(r_1^{7/2} \sqrt{r_2} + \sqrt{r_1} r_2^{7/2})}{(\sqrt{r_1} + \sqrt{r_2})^4} \\ &> \frac{r_1^4 + r_2^4 + 8(r_1^2 r_2^2 + r_1^3 r_2 + r_1^{5/2} r_2^{3/2} + r_1^{3/2} r_2^{5/2} + r_1 r_2^3) + 4(r_1^{7/2} \sqrt{r_2} + \sqrt{r_1} r_2^{7/2})}{(\sqrt{r_1} + \sqrt{r_2})^4} \\ &= \left( \frac{r_1 r_2}{(\sqrt{r_1} + \sqrt{r_2})^2} + r_1 \right)^2 + \left( \frac{r_1 r_2}{(\sqrt{r_1} + \sqrt{r_2})^2} + r_2 \right)^2 \\ &= (r + r_1)^2 + (r + r_2)^2. \end{aligned}$$

□

By Observation 4.3 we now know that for all the angles  $\theta_i$ , we have  $90^\circ < \theta_i < 180^\circ$ , and hence  $-1 < \cos \theta_i < 0$ . So in the parameterization of  $x_1$  and  $x_2$

$$x_1 = \frac{m_1^2 - n_1^2}{m_1^2 + n_1^2}, \quad x_2 = \frac{m_2^2 - n_2^2}{m_2^2 + n_2^2},$$

we must choose  $n_i > m_i$ . In this case  $x_3$  in (4) must satisfy  $(m_1^2 - n_1^2)(m_2^2 - n_2^2) - 4m_1m_2n_1n_2 < 0$ , which is equivalent to  $(m_1n_2 + m_2n_1)^2 > (m_1m_2 - n_1n_2)^2$ . Since  $m_i < n_i$  this is equivalent to  $m_1n_2 + m_2n_1 > n_1n_2 - m_1m_2$ , or equivalently

$$m_1n_2 + m_2n_1 + m_1m_2 > n_1n_2. \quad (8)$$

Looking at the expression for  $r_1$ ,

$$r_1 = \frac{n_1(m_1n_2 + m_2n_1)}{-m_1^2n_2 - n_1^2n_2 \pm (m_1n_1n_2 + m_2n_1^2)},$$

we see that in order for  $r_1 > 0$  to hold we must have  $n_1(m_1n_2 + m_2n_1) > n_2(m_1^2 + n_1^2)$ . Re-solving for the radii  $r_2$  and  $r_3$  using the positive term in the expression for  $r_1$ , we obtain:

$$r_2 = \frac{n_1n_2}{m_2n_1 + m_1n_2 - n_1n_2}, \quad r_3 = \frac{n_2(m_1n_2 + m_2n_1)}{n_1n_2^2 + m_2^2n_1 - m_1n_2^2 - m_2n_1n_2},$$

which give us two additional constraints in order to assure positive radii:  $m_2n_1 + m_1n_2 > n_1n_2$  and  $n_1(m_2^2 + n_2^2) > n_2(m_1n_2 + m_2n_1)$ . Note that the first of these constraints is stronger than (8), and so will replace it in the following summarizing theorem:

**Theorem 4.4** *Let  $m_1, n_1, m_2, n_2 \in \mathbb{N}$  that satisfy  $n_1 > m_1$ ,  $n_2 > m_2$ ,  $m_1n_2 + m_2n_1 > n_1n_2$ ,  $n_1(m_1n_2 + m_2n_1) > n_2(m_1^2 + n_1^2)$ , and  $n_1(m_2^2 + n_2^2) > n_2(m_1n_2 + m_2n_1)$ . Then all rational cosines  $x_i$  of a 3-petal flower are parametrized by:*

$$x_1 = \frac{m_1^2 - n_1^2}{m_1^2 + n_1^2}, \quad x_2 = \frac{m_2^2 - n_2^2}{m_2^2 + n_2^2}, \quad x_3 = \frac{(m_1^2 - n_1^2)(m_2^2 - n_2^2) - 4m_1m_2n_1n_2}{(m_1^2 + n_1^2)(m_2^2 + n_2^2)}.$$

*Assuming the center coin has radius one, then all the rational radii  $r_i$  of the outer coins are parametrized by:*

$$\begin{aligned} r_1 &= \frac{n_1(m_1n_2 + m_2n_1)}{m_1n_1n_2 + m_2n_1^2 - m_1^2n_2 - n_1^2n_2} \\ r_2 &= \frac{n_1n_2}{m_2n_1 + m_1n_2 - n_1n_2} \\ r_3 &= \frac{n_2(m_1n_2 + m_2n_1)}{n_1n_2^2 + m_2^2n_1 - m_1n_2^2 - m_2n_1n_2}. \end{aligned}$$

*This parameterization characterizes all sets of four mutually tangent Soddy circles of rational radius in the plane.*

EXAMPLE: Consider  $m_1 = 1$ ,  $n_1 = 2$ ,  $m_2 = 4$ , and  $n_2 = 5$ . We can see that the constraints will be satisfied, in particular the nontrivial ones  $m_1n_2 + m_2n_1 = 1 \cdot 5 + 4 \cdot 2 = 13 > 10 = 2 \cdot 5 = n_1n_2$ ,  $n_1(m_1n_2 + m_2n_1) = 2(1 \cdot 5 + 4 \cdot 2) = 26 > 25 = 5(1 + 4) = n_2(m_1^2 + n_1^2)$  and  $n_1(m_2^2 + n_2^2) = 2(16 + 25) = 82 > 65 = 5(1 \cdot 5 + 4 \cdot 2) = n_2(m_1n_2 + m_2n_1)$ . Then we have  $x_1 = -\frac{3}{5}$ ,  $x_2 = -\frac{9}{41}$ ,  $x_3 = -\frac{133}{205}$ , and the corresponding radii  $r_1 = 26$ ,  $r_2 = \frac{54}{11}$ ,  $r_3 = \frac{351}{59}$ . By scaling by the factor of  $\gcd(11, 59) = 649$  we obtain an integral flower with center radius of  $r = 649$  and the outer radii  $r_1 = 16874$ ,  $r_2 = 3186$  and  $r_3 = 3861$ .

## 4.2 Descartes' circle theorem and another parametrization

A nice relation connecting the radii of four mutually tangent Soddy circles in the Euclidean plane is given by Descartes' circle theorem [2].

**Theorem 4.5 (Descartes)** *A collection of four mutually tangent circles in the plane, where  $b_i = 1/r_i$  denotes the curvatures of the circles, satisfies the relation*

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = \frac{1}{2}(b_1 + b_2 + b_3 + b_4)^2.$$

Four mutually tangent circles in the plane are many times referred to as *Soddy circles* for Frederick Soddy, an English chemist who rediscovered Descartes' Circle Theorem in 1936 [2]. This theorem has also been generalized to higher dimensions.

It is straightforward to check that our rational parameterization from Theorem 4.4 satisfies Descartes' circle theorem. Another elegant parametrization of integer Soddy circles are given by Graham et al. in [5] in the following theorem.

**Theorem 4.6 (Graham et al.)** *The following parametrization characterizes the integral curvatures of a set of Soddy circles:*

$$b_1 = x, \quad b_2 = d_1 - x, \quad b_3 = d_2 - x, \quad b_4 = -2m + d_1 + d_2 - x,$$

where  $x^2 + m^2 = d_1 d_2$  and  $0 \leq 2m \leq d_1 \leq d_2$ .

We conclude this section by briefly comparing our rational parametrization to the one given by Theorem 4.6. Suppose we have a 3-petal flower, the coins of which have integer radii. Further, assume the center coin is the first one with curvature  $b_1$ . By scaling to make the center coin of radius one and conveniently permuting indices, the remaining outer coins have radii  $r_1, r_2, r_3$  given by

$$r_1 = \frac{b_1}{b_2}, \quad r_2 = \frac{b_1}{b_4}, \quad r_3 = \frac{b_1}{b_3}.$$

By Theorem 4.4 we have that

$$\begin{aligned} \frac{b_1}{b_2} &= \frac{n_1(m_1 n_2 + m_2 n_1)}{-m_1^2 n_2 - n_1^2 n_2 + m_1 n_1 n_2 + m_2 n_1^2} \\ \frac{b_1}{b_3} &= \frac{n_2(m_1 n_2 + m_2 n_1)}{-m_1 n_2^2 - m_2 n_1 n_2 + n_1 n_2^2 + m_2^2 n_1} \\ \frac{b_1}{b_4} &= \frac{n_1 n_2}{-n_1 n_2 + m_2 n_1 + m_1 n_2}. \end{aligned}$$

Replacing each  $b_i$  with the integer parametrization from Theorem 4.6 we can solve for  $d_1/x, d_2/x$  and  $m/x$  in terms of  $m_1, m_2, n_1, n_2$ , and obtain

$$\frac{m}{x} = \frac{(n_1 n_2 - m_1 m_2)}{m_1 n_2 + m_2 n_1}, \quad \frac{d_1}{x} = \frac{n_2(m_1^2 + n_1^2)}{n_1(m_1 n_2 + m_2 n_1)}, \quad \frac{d_2}{x} = \frac{n_1(m_2^2 + n_2^2)}{n_2(m_1 n_2 + m_2 n_1)}.$$

Hence,  $1 + (m/x)^2 = (d_1/x)(d_2/x)$  so the quadratic equation relating the parameters in Theorem 4.6 is satisfied.

The first inequality,  $0 \leq 2m$ , will clearly hold since when we choose  $m_i < n_i$  for  $i = 1, 2$ . Using the inequality constraints from Theorem 4.4 we obtain

$$\frac{d_1}{x} = \frac{n_2(m_1^2 + n_1^2)}{n_1(m_1n_2 + m_2n_1)} \leq \frac{n_1n_2(m_1n_2 + m_2n_1)}{n_1n_2(m_1n_2 + m_2n_1)} \leq \frac{n_1(m_2^2 + n_2^2)}{n_2(m_1n_2 + m_2n_1)} = \frac{d_2}{x},$$

and hence third inequality  $d_1 \leq d_2$  holds.

However, the second inequality,  $2m \leq d_1$  in Theorem 4.6, does not hold. In fact, one can show that the *opposite* inequality holds for all  $m_1, m_2, n_1, n_2$ , that satisfy the conditional inequalities given in Theorem 4.4. This does not mean there is anything wrong with the parametrization in either Theorem 4.4 or Theorem 4.6, since different range for parameters certainly can yield same solution set. Insisting the opposite  $d_1 \leq 2m$  in Theorem 4.6 might also yield all integer curvatures of Soddy circles.

Although equivalent, there is a subtle difference between presenting the integer radii of Soddy circles and presenting the integer curvatures. Suppose we have integer curvatures  $b_1, b_2, b_3$  and  $b_4$  of Soddy circles, and we would like to find the corresponding scaled configuration of Soddy circles with integer radii. Hence we are seeking  $r_1, r_2, r_3, r_4$  and  $N$  such that  $N/r_i = b_i$  for each  $i$ . As  $\text{lcm}(b_1, b_2, b_3, b_4)$  divides  $N$  we have that  $r_i = k \cdot \text{lcm}(b_1, b_2, b_3, b_4)/b_i$  for each  $i$ , where  $k$  is some positive integer. The other conversion, from integer radii to integer curvatures is similar.

In conclusion, we see that Graham et al.'s characterization of integer curvatures of Soddy circles in Theorem 4.6 is implied by our rational parametrization of the radii of 3-petal flowers in Theorem 4.4. In addition, the parametrization given by Graham et al in Theorem 4.6 relies on solving the Diophantine equation  $x^2 + m^2 = d_1d_2$  for each chosen  $x$ ,  $d_1$ , and  $d_2$ , while the parametrization developed here and given in Theorem 4.4 does not rely on satisfying any such equation, only inequalities.

## Acknowledgments

Sincere thanks to the anonymous referees for ...

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May 20, 2010

## A Generalizations of the Pythagorean Triples

In the following, a *primitive solution* is a solution where  $x, y$ , and  $z$  are pairwise relatively prime. To prove Theorem 4.1, we need the following:

**Claim A.1** *If  $r, s, t$  are positive integers such that  $r$  and  $s$  are relatively prime and  $rs = t^2$  then there are relatively prime integers  $m$  and  $n$  such that  $r = m^2$  and  $s = n^2$ .*

*Proof.* [Theorem 4.1] Note that  $\gcd(b, c) = 1$ . This proof follows and extends the exposition in [9]. Assume  $x, y, z$  form a primitive solution. In this case,  $x$  and  $y$  cannot both be even.

Case 1:  $x, y$  are both odd. Then  $x^2 \equiv 1 \pmod{4}$  and  $y^2 \equiv 1 \pmod{4}$ , giving  $z^2 \equiv 1 + \beta \pmod{4}$ . Since  $z^2 \equiv 0, 1 \pmod{4}$ , then  $\beta \equiv 0$  or  $\beta \equiv 3 \pmod{4}$  must hold. However,  $\beta \equiv 0 \pmod{4}$  implies that 4 divides  $\beta$ , contradicting the assumption that  $\beta$  is square-free. So the only case to consider here is the case where  $z$  is even and  $\beta \equiv 3 \pmod{4}$ .

$$\beta y^2 = z^2 - x^2 = (z + x)(z - x). \quad (9)$$

Letting  $\gcd(z + x, z - x) = d$  we get that  $d$  divides both  $z + x + z - x = 2z$  and  $z + x - (z - x) = 2x$ . Since  $x$  and  $z$  are relatively prime,  $d = 1$  or  $2$ . Since both  $z + x$  and  $z - x$  are odd, then  $d = 1$  must hold. Since now  $\gcd(z + x, z - x) = 1$  we have from (9) that for some factorization  $\beta = bc$  then  $r = z + x$  is divisible by  $b$  and  $s = z - x$  is divisible by  $c$ . Since  $\gcd\left(\frac{r}{b}, \frac{s}{c}\right) = 1$ , we have by Claim A.1 that  $m^2 = \frac{r}{b}$  and  $n^2 = \frac{s}{c}$ , and hence  $y = mn$ ,  $x = \frac{r-s}{2} = \frac{bm^2-cn^2}{2}$ , and  $z = \frac{r+s}{2} = \frac{bm^2+cn^2}{2}$ .

Case 2:  $x$  is even and  $y$  is odd. Then  $x^2 \equiv 0 \pmod{4}$  and  $y^2 \equiv 1 \pmod{4}$ , giving  $z^2 \equiv \beta \pmod{4}$ . Therefore  $\beta \equiv 0$  or  $\beta \equiv 1 \pmod{4}$ . However,  $\beta \equiv 0 \pmod{4}$  implies that 4 divides  $\beta$ ,

again contradicting the assumption that  $\beta$  is square-free. So the only case to consider here is the case where  $z$  is odd and  $\beta \equiv 1 \pmod{4}$ , which proceeds exactly as in case 1.

Case 3:  $x$  is odd and  $y$  is even. Then  $x^2 \equiv 1 \pmod{4}$  and  $y^2 \equiv 0 \pmod{4}$ , giving  $z^2 \equiv 1 \pmod{4}$ , and so  $z$  is odd.

Unlike cases 1 and 2,  $z + x$  and  $z - x$  are both even. Letting  $\gcd\left(\frac{z+x}{2}, \frac{z-x}{2}\right) = d$  we get that  $d$  divides  $\frac{z+x+z-x}{2} = z$  and  $\frac{z+x-(z-x)}{2} = x$ . Since  $x$  and  $z$  are relatively prime,  $d = 1$ . Now we have  $\frac{\beta y^2}{4} = rs$  where  $r = \frac{z+x}{2}$  and  $s = \frac{z-x}{2}$ . Hence  $b$  divides  $r$  and  $c$  divides  $s$  for some appropriate factorization  $\beta = bc$ . Since  $\gcd\left(\frac{r}{b}, \frac{s}{c}\right) = 1$ , so we have by Claim A.1 that  $m^2 = \frac{r}{b}$  and  $n^2 = \frac{s}{c}$  and hence  $y = 2mn$ ,  $x = r - s = bm^2 - cn^2$ , and  $z = r + s = bm^2 + cn^2$ .

For the other direction, first we show that  $x, y, z$  as given in cases 1 and 2 do form a solution:

$$\begin{aligned} x^2 + \beta y^2 &= \left(\frac{bm^2 - cn^2}{2}\right)^2 + \beta(mn)^2 \\ &= \frac{(bm^2)^2 - 2bm^2cn^2 + (cn^2)^2}{4} + \beta(mn)^2 \\ &= \frac{(bm^2)^2 + 2\beta m^2n^2 + (cn^2)^2}{4} \\ &= \left(\frac{bm^2 + cn^2}{2}\right)^2. \end{aligned}$$

Also for case 3 we get:

$$\begin{aligned} x^2 + \beta y^2 &= (bm^2 - cn^2)^2 + \beta(2mn)^2 \\ &= (bm^2)^2 - 2bm^2cn^2 + (cn^2)^2 + \beta(2mn)^2 \\ &= (bm^2)^2 + 2\beta m^2n^2 + (cn^2)^2 \\ &= (bm^2 + cn^2)^2. \end{aligned}$$

To show that the triple is primitive for cases 1 and 2, assume on the contrary that  $\gcd(x, y, z) = d > 1$ . Then there is a prime  $p$  that divides  $d$ . This  $p$  divides  $x$  and  $z$  and also their sum and difference:  $x + z = \frac{bm^2 - cn^2}{2} + \frac{bm^2 + cn^2}{2} = bm^2$  and  $x - z = \frac{bm^2 - cn^2}{2} - \frac{bm^2 + cn^2}{2} = -cn^2$ . This contradicts the assumption that  $bm^2$  and  $cn^2$  are relatively prime.

For case 3, again assume on the contrary that  $(x, y, z) = d > 1$ . Then there is an odd prime  $p$  that divides  $d$ .  $p \neq 2$  because  $x$  and  $z$  are both odd. This  $p$  divides  $x$  and  $z$  and also their sum and difference:  $x + z = 2bm^2$  and  $x - z = 2cn^2$ . Again, this contradicts the assumption that  $bm^2$  and  $cn^2$  are relatively prime.  $\square$



## B The polynomial $P_5$

$$\begin{aligned}
P_5 &= P_4(x_1, x_2, x_3, \text{EC}_2(x_4, x_5)) \cdot P_4(x_1, x_2, x_3, \overline{\text{EC}_2}(x_4, x_5)) \\
&= x_5^8 - 8x_1x_2x_3x_4x_5^7 - 8x_3^2x_4^2x_5^6 + 4x_2^2x_5^6 - 4x_5^6 + 4x_3^2x_5^6 + 16x_1^2x_2^2x_3^2x_5^6 \\
&\quad - 8x_2^2x_3^2x_5^6 - 8x_1^2x_4^2x_5^6 - 8x_1^2x_3^2x_5^6 - 8x_1^2x_2^2x_5^6 + 4x_4^2x_5^6 - 8x_2^2x_4^2x_5^6 \\
&\quad + 16x_1^2x_3^2x_4^2x_5^6 + 16x_2^2x_3^2x_4^2x_5^6 + 4x_1^2x_5^6 + 16x_1^2x_2^2x_4^2x_5^6 + 40x_1x_2^3x_3x_4x_5^5 \\
&\quad + 40x_1x_2x_3x_4^3x_5^5 - 32x_1^3x_2x_3x_4^3x_5^5 + 40x_1^3x_2x_3x_4x_5^5 - 32x_1^3x_2x_3^3x_4x_5^5 \\
&\quad - 32x_1x_2^3x_3^3x_4x_5^5 - 32x_1x_2^3x_3x_4^3x_5^5 - 24x_1x_2x_3x_4x_5^5 - 32x_1^3x_2^3x_3x_4x_5^5 \\
&\quad - 32x_1x_2x_3^3x_4x_5^5 + 40x_1x_2x_3^3x_4x_5^5 + 64x_1^2x_2^4x_3^2x_4^2x_5^4 - 16x_1^4x_4^2x_5^4 \\
&\quad + 28x_2^2x_4^2x_5^4 - 16x_3^2x_4^4x_5^4 - 24x_1^2x_2^2x_3^2x_4^4x_5^4 + 28x_1^2x_4^2x_5^4 - 12x_3^2x_5^4 \\
&\quad + 28x_2^2x_3^2x_4^4x_5^4 - 16x_2^2x_3^4x_4^4x_5^4 - 16x_2^2x_4^4x_5^4 + 64x_1^2x_2^2x_3^4x_4^2x_5^4 - 24x_2^2x_3^2x_4^2x_5^4 \\
&\quad + 16x_1^4x_4^4x_5^4 - 12x_4^4x_5^4 - 24x_1^2x_2^2x_3^2x_4^4x_5^4 + 6x_5^4 + 6x_4^4x_5^4 - 16x_1^2x_3^4x_5^4 \\
&\quad - 16x_1^2x_4^4x_5^4 + 6x_4^4x_5^4 + 64x_1^4x_2^2x_3^2x_4^2x_5^4 + 16x_1^4x_3^4x_5^4 + 16x_2^4x_3^4x_5^4 \\
&\quad - 16x_1^4x_2^2x_5^4 - 24x_1^2x_2^2x_4^2x_5^4 + 16x_1^4x_2^4x_5^4 + 16x_3^4x_4^4x_5^4 + 6x_4^4x_5^4 - 12x_2^2x_5^4 \\
&\quad - 144x_1^2x_2^2x_3^2x_4^2x_5^4 + 64x_1^2x_2^2x_3^2x_4^4x_5^4 + 28x_1^2x_3^2x_5^4 - 16x_2^4x_4^2x_5^4 \\
&\quad - 16x_2^4x_3^2x_5^4 - 16x_3^4x_4^2x_5^4 + 28x_2^2x_4^2x_5^4 + 28x_1^2x_2^2x_5^4 - 12x_1^2x_5^4 - 16x_1^4x_3^2x_4^4x_5^4 \\
&\quad + 6x_1^4x_5^4 - 16x_1^2x_2^4x_5^4 + 16x_2^4x_4^4x_5^4 + 112x_1^3x_2x_3^3x_4x_5^3 + 112x_1^3x_2x_3x_4^3x_5^3 \\
&\quad + 40x_1x_2x_3^5x_4x_5^3 - 32x_1^5x_2x_3x_4^3x_5^3 - 32x_1x_2^5x_3x_4^3x_5^3 + 112x_1^3x_2^3x_3x_4x_5^3 \\
&\quad + x_1x_2^5x_3x_4x_5^3 + 40x_1^5x_2x_3x_4x_5^3 + 40x_1x_2x_3x_4^5x_5^3 - 32x_1^3x_2x_3x_4^5x_5^3 \\
&\quad - 112x_1x_2^3x_3x_4x_5^3 - 32x_1x_2x_3^5x_4x_5^3 - 32x_1^5x_2x_3^3x_4x_5^3 - 112x_1^3x_2x_3x_4x_5^3 \\
&\quad + 112x_1x_2^3x_3^3x_4x_5^3 - 32x_1x_2^3x_3^5x_4x_5^3 - 32x_1x_2^5x_3^3x_4x_5^3 - 32x_1x_2^3x_3x_4^5x_5^3 \\
&\quad - 32x_1^3x_2^5x_3x_4x_5^3 + 112x_1x_2x_3^3x_4^3x_5^3 + 112x_1x_2^3x_3^3x_4^3x_5^3 - 112x_1x_2x_3^3x_4x_5^3 \\
&\quad - 112x_1x_2x_3x_4^3x_5^3 + 72x_1x_2x_3x_4x_5^3 - 128x_1^3x_2^3x_3^3x_4^3x_5^3 - 32x_1^5x_2^3x_3x_4x_5^3 \\
&\quad - 32x_1^3x_2x_3^5x_4x_5^3 - 32x_1x_2x_3^3x_4^5x_5^3 + 16x_1^2x_2^6x_4^2x_5^2 + 28x_2^2x_4^4x_5^2 + 28x_1^4x_2^2x_3^2x_5^2 \\
&\quad - 8x_2^2x_4^6x_5^2 - 12x_4^4x_5^2 - 16x_1^4x_4^4x_5^2 + 16x_1^2x_3^2x_4^6x_5^2 - 24x_1^2x_2^2x_3^4x_5^2 - 8x_1^6x_3^2x_5^2 \\
&\quad - 32x_2^2x_3^2x_5^2 - 24x_2^2x_3^2x_4^4x_5^2 - 16x_1^4x_2^4x_5^2 - 8x_1^6x_2^4x_5^2 + 40x_2^2x_3^2x_4^2x_5^2 \\
&\quad - 16x_1^4x_3^4x_5^2 - 24x_1^4x_2^2x_4^2x_5^2 + 28x_3^4x_4^2x_5^2 - 8x_1^2x_2^6x_5^2 + 64x_1^2x_2^4x_3^4x_4^2x_5^2 \\
&\quad - 32x_1^2x_4^2x_5^2 - 16x_3^4x_4^4x_5^2 - 12x_4^4x_5^2 + 28x_2^2x_3^4x_5^2 - 144x_1^2x_2^2x_3^4x_4^2x_5^2 \\
&\quad + 16x_2^2x_3^6x_4^2x_5^2 + 28x_1^2x_4^4x_5^2 + 4x_4^6x_5^2 + 64x_1^2x_2^2x_3^4x_4^4x_5^2 - 24x_2^4x_3^2x_4^2x_5^2 \\
&\quad + 28x_2^2x_3^2x_5^2 - 8x_1^6x_2^2x_5^2 + 16x_1^6x_2^2x_3^2x_5^2 + 16x_1^2x_2^2x_4^6x_5^2 + 16x_2^2x_3^2x_4^6x_5^2 \\
&\quad + 12x_2^2x_5^2 - 24x_1^4x_2^2x_3^2x_5^2 + 16x_1^2x_3^6x_4^2x_5^2 - 24x_1^2x_4^4x_2^2x_5^2 - 8x_2^2x_3^6x_5^2 \\
&\quad + 64x_1^4x_2^2x_3^2x_4^4x_5^2 + 192x_1^2x_2^2x_3^2x_4^2x_5^2 - 12x_2^4x_5^2 - 24x_1^2x_4^4x_2^2x_5^2 + 12x_2^2x_5^2 \\
&\quad - 8x_1^2x_3^6x_5^2 + 40x_1^2x_3^2x_4^2x_5^2 - 24x_1^2x_2^2x_4^4x_5^2 - 32x_1^2x_2^2x_5^2 + 64x_1^4x_2^2x_3^4x_4^2x_5^2 \\
&\quad + 28x_2^2x_4^4x_5^2 - 8x_1^2x_4^6x_5^2 - 4x_5^2 + 4x_1^6x_5^2 + 12x_1^2x_5^2 + 28x_1^4x_4^2x_5^2 - 16x_2^4x_3^4x_5^2 \\
&\quad + 16x_2^6x_3^2x_4^2x_5^2 - 8x_2^6x_4^2x_5^2 + 16x_1^6x_2^2x_4^2x_5^2 + 64x_1^4x_2^4x_3^2x_4^2x_5^2 \\
&\quad - 24x_1^2x_3^2x_4^4x_5^2 + 12x_2^2x_5^2 + 16x_1^2x_2^2x_3^6x_5^2 + 16x_2^6x_3^2x_4^2x_5^2 + 16x_1^2x_6^2x_3^2x_5^2 \\
&\quad + 40x_1^2x_2^2x_4^2x_5^2 - 8x_3^2x_4^6x_5^2 - 24x_1^2x_3^4x_4^2x_5^2 - 16x_2^4x_4^4x_5^2 + 28x_1^2x_3^4x_5^2
\end{aligned}$$

$$\begin{aligned}
& - 144x_1^2x_2^2x_3^2x_4^2x_5^2 + 28x_2^4x_4^2x_5^2 - 8x_3^6x_4^2x_5^2 - 32x_2^2x_4^2x_5^2 + 64x_1^2x_2^4x_3^2x_4^2x_5^2 \\
& + 40x_1^2x_2^2x_3^2x_5^2 - 24x_2^2x_3^4x_4^2x_5^2 - 144x_1^2x_2^4x_3^2x_4^2x_5^2 - 12x_1^4x_5^2 + 4x_3^6x_5^2 \\
& + 28x_1^2x_2^4x_5^2 - 144x_1^4x_2^2x_3^2x_4^2x_5^2 - 8x_2^6x_3^2x_5^2 + 28x_1^4x_2^2x_5^2 - 32x_3^2x_4^2x_5^2 \\
& - 32x_1^2x_3^2x_5^2 + 4x_2^6x_5^2 - 24x_1^4x_2^2x_3^2x_4^2x_5^2 - 24x_1x_2^5x_3x_4x_5 - 112x_1^3x_2x_3x_4^3x_5 \\
& - 112x_1^3x_2^3x_3x_4x_5 + 40x_1^3x_2x_3^5x_4x_5 - 112x_1x_2x_3^3x_4^3x_5 - 32x_1x_2^3x_3^5x_4^3x_5 \\
& - 32x_1^3x_2x_3^5x_4^3x_5 - 8x_1x_2x_3^7x_4x_5 + 40x_1^3x_2x_3x_4^5x_5 - 24x_1x_2x_3x_4^5x_5 \\
& - 112x_1x_2^3x_3x_4^3x_5 + 40x_1x_2^5x_3^3x_4x_5 + 40x_1x_2^3x_3x_4^5x_5 - 32x_1^5x_2x_3^3x_4^3x_5 \\
& - 32x_1^3x_2^5x_3^3x_4x_5 + 40x_1^3x_2^5x_3x_4x_5 - 8x_1x_2^7x_3x_4x_5 - 8x_1x_2x_3x_4^7x_5 + 112x_1x_2^3x_3^3x_4^3x_5 \\
& + 40x_1x_2^5x_3x_4^3x_5 - 112x_1^3x_2x_3^3x_4x_5 + 40x_1^5x_2x_3^3x_4x_5 + 72x_1x_2x_3^3x_4x_5 \\
& - 32x_1x_2^3x_3^3x_4^5x_5 + 72x_1x_2^3x_3x_4x_5 + 40x_1^5x_2^3x_3x_4x_5 + 40x_1^5x_2x_3x_4^3x_5 \\
& - 32x_1^3x_2^3x_3x_4^5x_5 - 24x_1x_2x_3^5x_4x_5 - 32x_1^5x_2^3x_3x_4^3x_5 - 32x_1^3x_2^5x_3x_4^3x_5 \\
& - 32x_1x_2^5x_3^3x_4^3x_5 + 112x_1^3x_2^3x_3x_4^3x_5 + 40x_1x_2x_3^5x_4^3x_5 - 40x_1x_2x_3x_4x_5 \\
& + 40x_1x_2x_3^3x_4^5x_5 - 32x_1^5x_2^3x_3^3x_4x_5 - 112x_1x_2^3x_3^3x_4x_5 - 24x_1^5x_2x_3x_4x_5 \\
& - 32x_1^3x_2^5x_3^3x_4x_5 + 72x_1x_2x_3x_4^3x_5 - 8x_1^7x_2x_3x_4x_5 + 112x_1^3x_2^3x_3^3x_4x_5 \\
& + 112x_1^3x_2x_3^3x_4^3x_5 - 32x_1^3x_2x_3^3x_4^5x_5 + 40x_1x_2^3x_3^5x_4x_5 + 72x_1^3x_2x_3x_4x_5 \\
& + 28x_1^4x_2^3x_4^2 + 16x_1^2x_2^6x_3^2x_4^2 - 24x_1^2x_2^2x_3^2x_4^4 + 28x_1^2x_2^2x_4^4 - 8x_1^2x_3^2x_4^6 \\
& - 16x_1^2x_4^4x_4^2 + 28x_2^2x_3^4x_4^2 + 28x_1^4x_2^2x_4^2 - 32x_1^2x_2^2x_4^2 - 8x_1^2x_2^2x_4^6 + 4x_3^2x_4^6 \\
& - 24x_1^2x_2^2x_3^4x_4^2 - 12x_3^2x_4^4 + 16x_2^4x_3^4x_4^4 + 16x_1^2x_2^2x_3^6x_4^2 + 4x_2^2x_4^6 - 32x_1^2x_3^2x_4^2 \\
& + 4x_2^2x_3^6 - 24x_1^2x_2^4x_3^2x_4^2 - 12x_2^4x_4^2 + 28x_1^2x_3^4x_4^2 + 6x_1^4x_4^4 - 16x_1^4x_3^2x_4^4 + 28x_1^2x_2^4x_4^2 \\
& + 12x_1^2x_3^2 + 6x_3^4x_4^4 + 16x_1^4x_2^4x_4^4 - 8x_2^2x_3^2x_4^6 - 12x_2^2x_4^4 - 32x_2^2x_3^2x_4^2 - 8x_2^2x_3^6x_4^2 \\
& + 12x_2^2x_4^4 + 4x_2^6x_3^2 - 16x_1^4x_2^2x_3^4 - 16x_2^4x_3^4x_4^2 + 40x_1^2x_2^2x_3^2x_4^2 + 6x_3^4 - 8x_2^6x_3^2x_4^2 \\
& - 4x_2^2 - 16x_1^2x_3^4x_4^4 + x_3^8 - 12x_1^2x_4^4 - 16x_2^4x_3^2x_4^4 + 6x_1^4x_3^4 - 16x_1^4x_2^4x_4^2 - 12x_1^4x_2^2 \\
& + 16x_1^4x_3^4x_4^4 - 4x_4^2 - 8x_1^2x_3^6x_4^2 + x_4^8 - 8x_1^2x_2^6x_3^2 - 16x_1^4x_3^4x_4^2 + 28x_1^2x_2^2x_4^2 \\
& + 4x_3^6x_4^2 + 16x_1^4x_2^4x_3^4 + 16x_1^6x_2^2x_3^2x_4^2 + 12x_2^2x_3^2 + 4x_1^2x_3^6 + 4x_1^2x_4^6 + 4x_2^6x_4^2 \\
& - 8x_1^2x_2^2x_3^6 + 16x_1^2x_2^2x_3^6x_4^2 - 4x_4^6 - 8x_1^6x_2^2x_3^2 - 12x_1^2x_4^4 - 16x_1^4x_2^4x_3^2 - 12x_1^2x_3^4 \\
& - 12x_2^4x_3^2 - 16x_2^2x_3^4x_4^4 + 12x_1^2x_4^2 + x_1^8 + 4x_1^6x_4^2 - 24x_1^4x_2^2x_3^2x_4^2 - 8x_1^2x_2^6x_4^2 \\
& + 6x_4^4 + 12x_2^2x_4^2 + 28x_2^2x_3^2x_4^4 + 6x_1^4x_2^4 + 6x_1^4 + 28x_1^4x_2^2x_3^2 + 28x_2^4x_3^2x_4^2 + 6x_2^4x_3^4 \\
& - 32x_1^2x_2^2x_3^2 + 4x_1^2x_2^6 - 4x_3^2 - 4x_1^6 - 4x_1^2 - 8x_1^6x_2^2x_4^2 + x_2^8 - 16x_1^4x_2^2x_4^4 - 16x_1^2x_2^4x_3^4 \\
& + 4x_1^6x_2^2 + 6x_2^4x_4^4 - 4x_3^6 - 8x_1^6x_2^2x_4^2 - 12x_3^4x_4^2 + 12x_1^2x_2^2 - 12x_2^2x_3^4 + 28x_1^2x_3^2x_4^4 \\
& - 12x_1^4x_4^2 + 28x_1^2x_2^4x_3^2 + 1 - 4x_2^6 - 12x_1^4x_3^2 + 6x_2^4 + 4x_1^6x_3^2.
\end{aligned}$$